FINAL EXAM: SOLUTIONS - MATH 111
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Problem 1 *Find the point of intersection of the line that goes through the points* $(0, 2)$ *and* $(2, 1)$ *and of the line that is perpendicular to it and goes through* $(2, 5)$.

Solution:

The slope of the line passing through $(0, 2)$ and $(2, 1)$ is $m = \frac{1-2}{2-0} = -\frac{1}{2}$. Thus, its equation is given by $y - 2 = -\frac{1}{2}x$ i.e., $y = -\frac{1}{2}x + 2$. The line that is perpendicular to it and passes through $(2, 5)$ has slope 2 and equation $y - 5 = 2(x - 2)$, i.e., $y = 2x + 1$. The point of intersection of these two lines is found by setting $2x + 1 = -\frac{1}{2}x + 2$. This gives $\frac{5}{2}x = 1$, i.e., $x = \frac{2}{5}$. Therefore $y = 2\frac{2}{5} + 1$. The point of intersection is the point $(\frac{2}{5}, \frac{9}{5})$. ■

Problem 2 *Find the domain of the function* $f(x) = \sqrt{-x^2 + 7x - 10}$.

Solution:

We have $-x^2 + 7x - 10 \geq 0$ whence $-(x^2 - 7x + 10) \geq 0$, i.e., $x^2 - 7x + 10 \leq 0$. Thus $(x - 5)(x - 2) \leq 0$. Now create the sign table to find $2 \leq x \leq 5$. Thus $D(f) = \{x \in \mathbf{R} : 2 \leq x \leq 5\}$. ■

Problem 3 *Find the vertex, the opening direction, the* x*- and* y*-intercepts and sketch the graph of* $f(x) = -3x^2 + 6x$.

Solution:

The vertex is at $V = (-\frac{b}{2a}, f(-\frac{b}{2a})) = (1, 3)$. The parabola opens down, the x-intercepts are $(0, 0)$ and $(2, 0)$ and the y-intercept is $(0, 0)$. The graph would have been sketched here. ■

Problem 4 *Find the equation of the parabola that has vertex* $V = (4, -1)$ *and goes through the point* $(0, -2)$.

Solution:

The equation should be of the form $y = a(x - 4)^2 - 1$, whence, since the parabola goes through $(0, -2)$, we must have $-2 = a(0 - 4)^2 - 1$ whence $-2 = 16a - 1$ i.e., $16a = -1$, which gives $a = -\frac{1}{16}$. Therefore, the parabola has equation $y = -\frac{1}{16}(x - 4)^2 - 1$. ■

Problem 5 *Solve the equations*

1. $3^{x^2-9} = 81^{2x}$.

2. $\log_2(x + 5) - \log_2(x - 1) = 2$.

Solution:

We have $3^{x^2-9} = 81^{2x}$ implies $3^{x^2-9} = (3^4)^{2x}$, whence $3^{x^2-9} = 3^{8x}$. Therefore $x^2 - 9 = 8x$, i.e., $x^2 - 8x - 9 = 0$. This yields $(x - 9)(x + 1) = 0$, i.e., $x = -1$ or $x = 9$.

For the second part $\log_2(x + 5) - \log_2(x - 1) = 2$ implies $\log_2 \frac{x+5}{x-1} = \log_2 4$. Therefore $\frac{x+5}{x-1} = 4$, whence $x + 5 = 4(x - 1)$, which yields $x + 5 = 4x - 4$, i.e., $3x = 9$, or $x = 3$. This is an acceptable solution. ■

Problem 6 *Solve the following system* **by the Gauss-Jordan method**

$$\left\{\begin{array}{rcrcrcr} x & + & y & + & z & = & 0 \\ -2x & - & y & + & z & = & 5 \\ x & + & 2y & - & 2z & = & -1 \end{array}\right\}.$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ -2 & -1 & 1 & 5 \\ 1 & 2 & -2 & -1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & -3 & -1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -6 & -6 \end{array}\right] \to$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

■

Problem 7 *An urn contains 5 red, 3 black, 7 white and 2 green marbles. Two marbles are drawn at random without replacement. Find the probability of*

1. *the first marble being white and the second being green,*
2. *one marble being black and one red.*

Solution:

$$P(1W \cap 2G) = P(1W)P(2G|1W) = \frac{7}{17}\frac{2}{16}.$$

$$\begin{aligned} P(B \cap R) &= P(1B \cap 2R) + P(1R \cap 2B) \\ &= P(1B)P(2R|1B) + P(1R)P(2B|1R) \\ &= \tfrac{3}{17}\tfrac{5}{16} + \tfrac{5}{17}\tfrac{3}{16}. \end{aligned}$$

■

Problem 8 *In a U.S. state, 20% of the population lives in inner cities, 35% in suburbs and 45% in rural areas. 20% of those living in inner cities receive poor medical care and the corresponding probabilities for those living in the suburbs and in rural areas are 5% and 10%, respectively. A person in the population selected at random receives poor medical care. What is the probability that he came from the suburbs?*

Solution:

Let I, S, R and P denote inner cities, suburbs, rural areas and poor care, respectively. Then

$$\begin{aligned} P(S|P) &= \tfrac{P(P|S)P(S)}{P(P|I)P(I)+P(P|S)P(S)+P(P|R)P(R)} \\ &= \tfrac{0.05\cdot 0.35}{0.2\cdot 0.2+0.05\cdot 0.35+0.1\cdot 0.45}. \end{aligned}$$

■

Problem 9 *A committee of the United Nations consists of 6 Chinese, 5 Indian, 3 American, 2 Canadian and 4 European members.*

1. *A subcommittee of 5 is to be formed on Asian affairs. In how many ways can such a subcommittee be formed if it is to consist of at least 4 Asian members?*
2. *A subcommittee of 8 is to be formed consisting of a Chairman, a Vice-Chairman, two Secretaries and 4 members. In how many ways can such a subcommittee be formed?*

Solution:

1. $\binom{11}{4}\binom{9}{1} + \binom{11}{5}\binom{9}{0}$.
2. $20 \cdot 19 \cdot \binom{18}{2} \cdot \binom{16}{4}$.

■

Problem 10 *A pair of fair dice are rolled 9 times. Find the probabilities that*

1. *sum 8 appears at least once.*
2. *sum 10 appears at most twice.*

Solution:

1. Sum 8 has probability $\frac{5}{36}$ in a single roll. Thus

$$1 - \binom{9}{0}(\frac{5}{36})^0(\frac{31}{36})^9$$

is the probability of getting at least once a sum of 8.

2. Sum 10 has probability $\frac{1}{12}$, whence

$$\binom{9}{0}(\frac{1}{12})^0(\frac{11}{12})^9 + \binom{9}{1}(\frac{1}{12})^1(\frac{11}{12})^8 + \binom{9}{2}(\frac{1}{12})^2(\frac{11}{12})^7$$

is the probability of obtaining at most two sum 10's.

■