HOMEWORK 10: SOLUTIONS - MATH 111
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Problem 1 *In Canada, postal codes are made up of two triples: the first consist of a letter followed by a number followed by a letter, and the second triple by a number followed by a letter followed by a number. For instance H3A 2T5 is a valid Canadian postal code.*

1. *How many Canadian codes are possible if the beginning number of the second triple is arbitrary?*
2. *How many are possible if the beginning number of the second triple cannot be 0?*

Solution:

Partition the task of forming a Canadian code to six subtasks, all of which have to be performed in order for the entire task to be complete. The first subtask consists of choosing the first symbol, the second subtask of choosing the second symbol, and so on, until the sixth subtask consisting of choosing the last symbol.

1. Subtasks 1,3 and 5 may be accomplished in 26 ways each whereas subtasks 2,4 and 6 may be accomplished in 10 ways each. Thus, by the multiplication principle, one may form a postal code in $26^3 \cdot 10^3$ ways, i.e., there are $26^3 \cdot 10^3$ Canadian postal codes possible.
2. The only difference from part 1 is that one has only 9 instead of 10 choices for subtask 4. Hence, there are $26^3 \cdot 10^2 \cdot 9$ codes in which the beginning number of the second triple cannot be a 0.

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Problem 2 *How many different "words" may formed by using all the letters in the word "COMMITTEE"?*

Solution:

The word "COMMITTEE" contains 9 letters partitioned in 6 groups of Cs, Os, Is, Ms, Ts and Es of 1,1,1,2,2 and 2 letters, respectively. Therefore the number of different words that may be formed by using these letters is $\frac{9!}{(2!)^3}$. ■

Problem 3 *The U.S. senate has 51 republican and 49 democratic senators. A committee of 9 members is to be formed consisting of 5 republicans and 4 democratic senators. In how many ways is it possible to form such a committee?*

Solution:

Partition the task of forming the subcommittee of 5 republicans and 4 democrats in two subtasks. In subtask 1 the 5 republicans are chosen out of the 51 republican senators. In subtask 2 the committee is completed by choosing the 4 democratic members among the 49 democratic senators. The first subtask may be accomplished in $\binom{51}{5}$ ways whereas the second subtask may be accomplished in $\binom{49}{4}$ ways. Therefore, by the multiplication principle, there are $\binom{51}{5}\binom{49}{4}$ ways of forming the committee consisting of 5 republicans and 4 democrats. ■

Problem 4 *A bridge hand consists of 13 cards out of a normal deck of 52 cards. Find the probability that a bridge hand contains*

1. *7 face cards.*

2. *6 cards of one suit and 7 of another.*

Solution:

1. Partition the task of choosing a bridge hand containing 7 face and 6 non-face cards to two subtasks. In subtask 1 we choose the 7 face cards out of the 12 face cards available in the deck and in the second subtask we choose the remaining 6 non-face cards out of the 40 non-face cards that are available in the deck. The first subtask may be accomplished in $\binom{12}{7}$ different ways and the second subtask in $\binom{40}{6}$ different ways. Thus, by the multiplication principle, there are $\binom{12}{7}\binom{40}{6}$ different bridge hands containing 7 face cards and 6 non-face cards.

2. We work similarly as in 1. The entire task of choosing 6 cards of one suit and 7 of another is partitioned in the following 4 subtasks: in subtask 1 we choose the two out of the 4 suits, in subtask 2 we order the two chosen suits, in subtask 3 we choose 6 cards from the first chosen suit and in subtask 4 we choose the remaining 7 cards from the second chosen suit. Subtask 1 may be accomplished in $\binom{4}{2}$ ways, subtask 2 in 2 ways, aubtask 3 in $\binom{13}{6}$ ways and subtask 4 in $\binom{13}{7}$ ways. Therefore, by the multiplication principle, the entire task may be accomplished in $2\binom{4}{2}\binom{13}{6}\binom{13}{7}$ ways. I.e., there are that many bridge hands with 6 cards of one suit and 7 of another.

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Problem 5 *Suppose that a government agency has a board consisting of 6 Caucasian, 5 Hispanic and 4 African American members. A committee of 3 members of this board is to be formed to deal with issues concerning Hispanics. In how many ways can such a committee be formed so that at least one of the Hispanic board members is also a member of the committee?*

Solution:

The number of ways that a committee can be formed containing at least one Hispanic member equals the total number of 3-member committees minus the number of committees that contain no Hispanics by the difference rule. The total number of committees equals the number of ways of choosing 3 out of the total number of 15 board members, i.e., $\binom{15}{3}$. On the other hand, the number of committees containing no Hispanic members equals the number of ways of choosing all 3 members out of the 10 non-Hispanic members of the board, i.e., $\binom{10}{3}$. Therefore, the number of committees containing at least one Hispanic member is $\binom{15}{3} - \binom{10}{3}$. ■

Problem 6 *Given 4 balls, of which some are blue, pick 2 at random. The probability that both are blue is $\frac{1}{2}$. Determine how many of the 4 were blue.*

Solution:

Let n be the number of blue balls. We must have $\frac{\binom{n}{2}}{\binom{4}{2}} = \frac{1}{2}$. Thus $\frac{\frac{n!}{2!(n-2)!}}{\frac{4!}{2!2!}} = \frac{1}{2}$. Therefore $\frac{\frac{n(n-1)}{2}}{6} = \frac{1}{2}$. This yields $n(n-1) = 6$, whence $n^2 - n - 6 = 0$, i.e., $(n-3)(n+2) = 0$. Thus $n = -2$ or $n = 3$. Since the number of blue balls must be positive, we have $n = 3$. ■

Problem 7 *A coin is tossed 7 times. What is the probability of obtaining at least 6 heads? What is the probability of no more than 2 tails?*

Solution:

$$P(\geq 6H) = \binom{7}{6}(\frac{1}{2})^7 + \binom{7}{7}(\frac{1}{2})^7.$$

$$P(\leq 2T) = \binom{7}{0}(\frac{1}{2})^7 + \binom{7}{1}(\frac{1}{2})^7 + \binom{7}{2}(\frac{1}{2})^7.$$

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Problem 8 *A certain machine produces a defective item with probability 0.05. What is the probability that out of 100 items manufactured by this machine at least one defective item is produced?*

Solution:

$$P(\geq 1D) = 1 - P(0D) = 1 - \binom{100}{0}(0.05)^0(0.95)^{100}.$$

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