EXAM 2 - MATH 215

Wednesday, November 19, 2003
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Read each problem very carefully before starting to solve it. Each question is worth 4 points. It is necessary to show your work. Correct answers without explanations are worth 0 points.

GOOD LUCK!!

1. Prove or provide a counterexample for each of the statements: (Please, *state clearly* in each case whether you are proving or providing a counterexample.)
   (a) If $C \subseteq A$ and $D \subseteq B$, then $D - A \subseteq B - C$.
   (b) $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.

2. (a) i. Find the union and the intersection of the indexed collection $\mathcal{C} = \{C_n : n \in \mathbf{Z}\}$, where $C_n = [n, n+1)$, for all $n \in \mathbf{Z}$.
   ii. Let $\mathbb{R}$ be the universe of discourse. Give an example of a family $\mathcal{C}$ of pairwise disjoint subsets of $\mathbb{R}$, such that $\mathcal{C}$ has 5 elements and $\bigcup_{C \in \mathcal{C}} C = \mathbb{R}$.
   (b) Let $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ be a family of sets and B be a set. Prove that $B - \bigcup_{\alpha \in \Delta} A_\alpha = \bigcap_{\alpha \in \Delta}(B - A_\alpha)$.

3. Use the Principle of Mathematical Induction to prove the following:
   (a) $\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 14} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$, for all $n \geq 1$.
   (b) $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$, for all $n \geq 1$.

4. (a) Use the Principle of Mathematical Induction to show that $7^n - 4^n$ is divisible by 3, for every $n \geq 1$.
   (b) Recall from calculus the Quotient Rule for differentiation: $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$. The Quotient Rule yields immediately the Inverse Rule: $(\frac{1}{f})' = -\frac{f'}{f^2}$. Use the Inverse Rule combined with the Principle of Mathematical Induction to show that, if $f(x) = \frac{1}{x}$, then $f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}$, for all $n \geq 1$.

5. (a) Among 12 lottery finalists, four will be selected at random for individual prizes of $2,000, one for the Big Prize of a week-long vacation for two in Greece and one for the Huge Prize of a Lamborghini *Special Limited Edition*®. In how many ways may the prizes be distributed?
   (b) 9 American astronauts and 7 Russian cosmonauts have been training as candidates for a space mission. The final mission is to include a Russian and an American co-captain, 3 astronauts and 2 cosmonauts. How many possible selections can be made?