Problem 1 True or false?
(a) \([2, 5] = \{2, 3, 4, 5\}\)  (b) \([6, 9] \subseteq [6, 10]\)  (c) \(\{\emptyset\} \subseteq \emptyset, \{\emptyset\}\)
(d) \(\{1, 2\} \in \{\{1, 2, 3\}, \{1, 3\}, 1, 2\}\)  (e) \(\{4\} \subseteq \{1, 2, 3, \{4\}\}\).

Solution: (a) False, 2.5 \in [2, 5] but 2.5 \notin \{2, 3, 4, 5\}.
(b) This is true, since if \(6 < x < 9\), then \(6 \leq x < 10\).
(c) This is true since \(\emptyset \in \emptyset, \{\emptyset\}\).
(d) False, since the set \(\{1, 2\}\) is not an element of the set \(\{\{1, 2, 3\}, \{1, 3\}, 1, 2\}\).
(e) This is true, since \(\{4\} \in \{1, 2, 3, \{4\}\}\).  

Problem 2 Give an example, if there is one, of sets \(A, B\) and \(C\) such that the following are true. If there is no example write “Not possible”.
(a) \(A \subseteq B, B \subseteq C\) and \(C \subseteq A\)  (b) \(A \subseteq B, B \not\subseteq C\) and \(A \not\subseteq C\).

Solution: (a) Let \(A = B = C = \emptyset\).
(b) Let \(A = B = \{\emptyset\}\) and \(C = \emptyset\).  

Problem 3 Write the power set \(\mathcal{P}(X)\) for each of the sets (a) \(X = \{S, \{S\}\}\)  (b) \(X = \{1, \{2, \{3\}\}\}\).

Solution: (a) \(\mathcal{P}(X) = \{\emptyset, \{S\}, \{\{S\}\}, \{S, \{S\}\}\}\).
(b) \(\mathcal{P}(X) = \{\emptyset, \{1\}, \{\{2, \{3\}\}\}, \{1, \{2, \{3\}\}\}\}\).  

Problem 4 List all of the proper subsets for each of the following sets (a) \(\{\emptyset, \{\emptyset\}\}\)  (b) \(\{0, \Delta, \square\}\)

Solution: (a) \(\emptyset, \{\emptyset\}\).
(b) \(\emptyset, \{\emptyset\}, \{\Delta\}, \{\square\}, \{\emptyset, \Delta\}, \{\emptyset, \square\}, \{\Delta, \square\}\).  

Problem 5 Give an example, if there is one, of each of the following. If there is no example, write “Not possible”.
(a) A set \(A\) such that \(\mathcal{P}(A)\) has 64 elements.
(b) Sets \(A\) and \(B\) such that \(A \subseteq B\) and \(\mathcal{P}(B) \subseteq \mathcal{P}(A)\).
(c) A set \(A\) such that \(\mathcal{P}(A) = \emptyset\).
(d) Sets \(A, B\) and \(C\) such that \(A \subseteq B\), \(B \subseteq C\) and \(\mathcal{P}(A) \subseteq \mathcal{P}(C)\).

Solution: (a) \(A = \{0, 1, 2, 3, 4, 5\}\).
(b) \(A = B = \emptyset\).
(c) No such example exists because \(\emptyset \in \mathcal{P}(A)\) for every set \(A\).
(d) Set \(A = B = C = \emptyset\).  

Problem 6 Prove that if \(x \notin B\) and \(A \subseteq B\), then \(x \notin A\).

Solution: Suppose to the contrary that \(x \notin B, A \subseteq B\) and \(x \in A\). Since \(x \in A\) and \(A \subseteq B\), we get \(x \in B\). But this contradicts the assumption \(x \notin B\).
Problem 7 Let \( X = \{ x : P(x) \} \). Are the following statements true or false?
(a) If \( a \in X \), then \( P(a) \). (b) If \( P(a) \), then \( a \in X \). (c) If \( \sim P(a) \), then \( a \notin X \).

Solution: (a) This is true, by the definition of \( X \).
(b) This is also true by the definition of \( X \).
(c) This is also true since it is the contrapositive of part (a).  

Problem 8 Prove that \( X = Y \), where \( X = \{ x : x \in \mathbb{R} \text{ and } x \text{ is a solution to } x^2 - 7x + 12 = 0 \} \) and \( Y = \{3, 4\} \).

Solution: We first show that \( Y \subseteq X \). Suppose that \( x \in Y \). Then \( x = 3 \) or \( x = 4 \). Note that \( 3^2 - 7 \cdot 3 + 12 = 0 \) and \( 4^2 - 7 \cdot 4 + 12 = 0 \), whence \( x \in X \) in both cases, and, therefore \( Y \subseteq X \).

We now show that \( X \subseteq Y \). Suppose that \( x \in X \). Therefore \( x \in \mathbb{R} \) and \( x^2 - 7x + 12 = 0 \), whence \( (x - 3)(x - 4) = 0 \). Therefore \( x = 3 \) or \( x = 4 \) and, therefore \( x \in Y \). Thus, \( X \subseteq Y \).

Problem 9 Prove that \( X = Y \), where \( X = \{ x \in \mathbb{N} : x^2 < 30 \} \) and \( Y = \{1, 2, 3, 4, 5\} \).

Solution: First, we show that \( X \subseteq Y \). Suppose \( x \in X \). Then \( x \in \mathbb{N} \) and \( x^2 < 30 \). Thus \( x = 1, 2, 3, 4 \text{ or } 5 \). Therefore \( x \in Y \) and \( X \subseteq Y \).

Next, we show that, conversely, \( Y \subseteq X \). Suppose \( x \in Y \). Then \( x = 1, 2, 3, 4 \text{ or } 5 \). Note that \( 1, 2, 3, 4, 5 \in \mathbb{N} \text{ and } 1, 4, 9, 16, 25 < 30 \), whence \( x \in X \) and \( Y \subseteq X \).

Problem 10 Let the universe be all real numbers. Let \( A = [3, 8) \), \( B = [2, 6) \), \( C = (1, 4) \) and \( D = (5, \infty) \). Find \( B \cup C, A \cap B, D - A, \tilde{D} \text{ and } (A \cup C) - (B \cap D) \).

Solution: \( B \cup C = (1, 6], A \cap B = [3, 6], D - A = [8, \infty), \tilde{D} = (-\infty, 5] \text{ and } (A \cup C) - (B \cap D) = (1, 8) - (5, 6] = (1, 5] \cup (6, 8) \).

Problem 11 Let \( U = \{1, 2, 3\} \) be the universe for the sets \( A = \{1, 2\} \) and \( B = \{2, 3\} \). Find \( \mathcal{P}(A) \cap \mathcal{P}(B) \) and \( \mathcal{P}(A) - \mathcal{P}(B) \).

Solution: we have \( \mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \) and \( \mathcal{P}(B) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\} \). So \( \mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset, \{2\}\} \) and \( \mathcal{P}(A) - \mathcal{P}(B) = \{\{1\}, \{1, 2\}\} \).

Problem 12 Let \( A, B, C \) be sets.
(a) Prove that \( (A - B) - C = (A - C) - (B - C) \).
(b) Prove that if \( A \subseteq C \) and \( B \subseteq C \), then \( A \cup B \subseteq C \).

Solution: (a) We first show that \( (A - B) - C \subseteq (A - C) - (B - C) \).
Then \( x \in A - B \) and \( x \notin C \). Thus \( x \in A \) and \( x \notin B \) and \( x \notin C \).
Therefore \( x \in A \) and \( x \notin C \) and, at the same time \( x \notin B - C \subseteq B \).
Therefore \( x \in (A - C) - (B - C) \) and \( (A - B) - C \subseteq (A - C) - (B - C) \).
Next, we show that \((A - C) - (B - C) \subseteq (A - B) - C\). Suppose that \(x \in (A - C) - (B - C)\). Then \(x \in A - C\) but \(x \notin B - C\). Therefore \(x \in A\) and \(x \notin C\) and \((x \notin B\) or \(x \in C\)). Since \(x \notin C\), this last disjunction yields \(x \notin B\). Therefore \(x \in (A - B) - C\) and \((A - C) - (B - C) \subseteq (A - B) - C\).

(b) Suppose that \(A \subseteq C\) and \(B \subseteq C\) and let \(x \in A \cup B\). Then \(x \in A\) or \(x \in B\). If \(x \in A\), then, since \(A \subseteq C\), \(x \in C\). If \(x \in B\), then, since \(B \subseteq C\), we also get \(x \in C\). Therefore in each case, \(x \in C\), which yields that \(A \cup B \subseteq C\).

**Problem 13** Let \(A, B, C, D\) be sets. Prove that if \(A \cup B \subseteq C \cup D\), \(A \cap B = \emptyset\) and \(C \subseteq A\), then \(B \subseteq D\).

**Solution:** Suppose \(A \cup B \subseteq C \cup D\), \(A \cap B = \emptyset\) and \(C \subseteq A\). Then \(x \in A \cup B\), whence, since \(A \cup B \subseteq C \cup D\), \(x \in C \cup D\). Thus, either \(x \in C\) or \(x \in D\). If \(x \in C\), then, since \(C \subseteq A\), \(x \in A\), which together with \(x \in B\), contradicts \(A \cap B = \emptyset\). Therefore \(x \notin C\) and, hence \(x \in D\). Thus \(B \subseteq D\).

**Problem 14** Let \(A, B\) be sets. Prove that \(\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)\).

**Solution:** Suppose that \(X \in \mathcal{P}(A) \cup \mathcal{P}(B)\). Then \(X \in \mathcal{P}(A)\) or \(X \in \mathcal{P}(B)\). If \(X \in \mathcal{P}(A)\), then \(X \subseteq A \subseteq A \cup B\). Thus \(X \subseteq A \cup B\) and \(X \in \mathcal{P}(A \cup B)\). If, on the other hand, \(X \in \mathcal{P}(B)\), then \(X \subseteq B \subseteq A \cup B\), whence \(X \in \mathcal{P}(A \cup B)\) as well. Thus \(\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)\).

**Problem 15** Provide counterexamples for each of the following:

(a) If \(A \subseteq C \subseteq B \cap C\), then \(A \subseteq B\).

(b) \(A - (B - C) = (A - B) - C\).

**Solution:**

(a) Let \(A = \{0\}\) and \(B = C = \emptyset\). Then \(A \cap C = \emptyset \subseteq \emptyset = B \cap C\), but, obviously, \(A = \{0\} \notin \emptyset = B\.

(b) Set \(A = B = C = \{0\}\). Then \(A - (B - C) = \{0\}\), whereas \((A - B) - C = \emptyset\).

**Problem 16** Define the symmetric difference operation \(\triangle\) on sets by \(A \triangle B = (A - B) \cup (B - A)\). Prove that (a) \(A \triangle B = (A \cup B) - (A \cap B)\) and (b) \(A \triangle \emptyset = A\).

**Solution:**

(a) We first show that \((A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)\). So suppose \(x \in (A - B) \cup (B - A)\). Then \(x \in A - B\) or \(x \in B - A\). If \(x \in A - B\), then \(x \in \bar{B}\) and \(x \notin A\), whereas in the second case \(x \in B\) and \(x \notin A\), whence \(x \in B - A\). Thus, in both cases \(x \in (A - B) \cup (B - A)\).

(b) \(A \triangle \emptyset = \emptyset\).

**Problem 17** Find the union and intersection of each of the following families or indexed collections.

(a) Let \(\mathbb{R}^+ = (0, \infty)\). For \(r \in \mathbb{R}^+\), let \(A_r = [-\pi, r)\), and let \(A = \{A_r : r \in \mathbb{R}^+\}\).

(b) For each natural number \(n \geq 3\), let \(A_n = \left\lfloor \frac{1}{n}, 2 + \frac{1}{n} \right\rfloor\) and \(A = \{A_n : n \geq 3\}\).

(c) For each \(n \in \mathbb{N}\), let \(D_n = (-n, \frac{1}{n})\) and \(D = \{D_n : n \in \mathbb{N}\}\).
Problem 18 Let $A = \{A_\alpha : \alpha \in \Delta\}$ be a family of sets and let $B$ be a set. Prove that $B \cup \bigcap_{\alpha \in \Delta} A_\alpha = \bigcap_{\alpha \in \Delta} (B \cup A_\alpha)$.

Solution: We have
\[
x \in B \cup \bigcap_{\alpha \in \Delta} A_\alpha \iff \begin{cases} x \in B \text{ or } x \in \bigcap_{\alpha \in \Delta} A_\alpha \\ x \in B \text{ or } x \in A_\alpha, \text{ for all } \alpha \in \Delta \\ (x \in B \text{ or } x \in A_\alpha), \text{ for all } \alpha \in \Delta \\ x \in B \cup A_\alpha, \text{ for all } \alpha \in \Delta \\ x \in \bigcap_{\alpha \in \Delta} (B \cup A_\alpha) \end{cases}
\]

Problem 19 Let $\mathcal{A}$ be a family of sets, and suppose $\emptyset \in \mathcal{A}$. Prove that $\bigcap_{A \in \mathcal{A}} A = \emptyset$.

Solution: We show the contrapositive. Suppose that $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$. Then, there exists $x \in \bigcap_{A \in \mathcal{A}} A$. Therefore $x \in A$, for all $A \in \mathcal{A}$. Hence $A \neq \emptyset$, for all $A \in \mathcal{A}$. Thus, $\emptyset \notin \mathcal{A}$.

Problem 20 If $A = \{A_\alpha : \alpha \in \Delta\}$ is a family of sets and if $\Gamma \subseteq \Delta$, prove that $\bigcap_{\alpha \in \Gamma} A_\alpha \subseteq \bigcap_{\alpha \in \Delta} A_\alpha$.

Solution: Suppose that $\Gamma \subseteq \Delta$ and let $x \in \bigcap_{\alpha \in \Gamma} A_\alpha$. Then $x \in A_\alpha$, for all $\alpha \in \Gamma$. Thus, since $\Gamma \subseteq \Delta$, $x \in A_\alpha$, for all $\alpha \in \Delta$. Thus, $\bigcap_{\alpha \in \Gamma} A_\alpha \subseteq \bigcap_{\alpha \in \Delta} A_\alpha$.

Problem 21 Give an example of an indexed collection of sets $\{A_\alpha : \alpha \in \Delta\}$ such that each $A_\alpha \subseteq (0,1)$, and for all $\alpha$ and $\beta \in \Delta$, $A_\alpha \cap A_\beta \neq \emptyset$ but $\bigcap_{\alpha \in \Delta} A_\alpha = \emptyset$.

Solution: Suppose that $\Delta = \mathbb{N}$ and set $A_n = (0, \frac{1}{n})$, for all $n \in \mathbb{N}$. Then, for all $m, n \in \mathbb{N}$, $(0, \frac{1}{n}) \cap (0, \frac{1}{m}) = (0, \frac{1}{\max(m,n)}) \neq \emptyset$, whereas $\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset$.

Problem 22 Let $A$ and $B$ be two pairwise disjoint families of sets. Let $C = A \cap B$ and $D = A \cup B$.
(a) Prove that $C$ is a family of pairwise disjoint sets.
(b) Give an example to show that $D$ need not be pairwise disjoint.
(c) Prove that if $\bigcup_{A \in A} A$ and $\bigcup_{B \in B} B$ are disjoint, then $D$ is pairwise disjoint.

Solution: (a) Let $C_1, C_2 \in C$, then $C_1, C_2 \in A$ and $C_1, C_2 \in B$. Thus, since $A$ is pairwise disjoint, $C_1 \cap C_2 = \emptyset$, whence $C$ is also pairwise disjoint.
(b) Let $A = \{\{0\}\}, B = \{\{0, 1\}\}$. Obviously, both $A$ and $B$ are pairwise disjoint since they each contain only one set. But $D = A \cup B = \{\{0\}, \{0, 1\}\}$, which is not pairwise disjoint.
(c) We show the contrapositive. Suppose that $A \cup B$ is not pairwise disjoint. Therefore, there exist $A \in A$ and $B \in B$, such that $A \cap B \neq \emptyset$. Thus, there exist $x$, such that $x \in A \in A$ and $x \in B \in B$. But then $x \in \bigcup_{A \in A} A$ and $x \in \bigcup_{B \in B} B$, whence $\bigcup_{A \in A} A$ and $\bigcup_{B \in B} B$ are not disjoint.