HOMEWORK 1 SOLUTIONS - MATH 325
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Problem 1 *Prove SSA for right triangles: If* $\overline{AB} \cong \overline{DE}, \overline{AC} \cong \overline{DF}$ *and if* $\widehat{B} = 90^o$ *and* $\widehat{E} = 90^o$, *then* $\triangle ABC \cong \triangle DEF$.

Solution: Extend $\overline{BC}$ on the side of B and take G, such that $BG \cong EF$. Then $AB \cong DE, GB \cong EF$ and $\widehat{ABG} \cong \widehat{DEF}$, whence, by ASA, $\triangle ABG \cong \triangle DEF$. Thus $AG \cong AC$, whence $\widehat{DFE} \cong \widehat{AGB} \cong \widehat{ACB}$ and, therefore, $\triangle ABC \cong \triangle DEF$, by ASA. ∎

Problem 2 *Prove SAA: If* $\widehat{A} \cong \widehat{D}, \widehat{B} \cong \widehat{E}$ *and* $\overline{BC} \cong \overline{EF}$, *then* $\triangle ABC \cong \triangle DEF$.

Solution: If you assume that it has been proven that the sum of the angles of a triangle is 180^o, then this is very easy and it's been given in your book:

Since $\widehat{A} \cong \widehat{D}$ and $\widehat{B} \cong \widehat{E}$, we get $\widehat{C} \cong 180^o - \widehat{A} - \widehat{B} \cong 180^o - \widehat{D} - \widehat{E} \cong \widehat{F}$. Therefore, by ASA, $\triangle ABC \cong \triangle DEF$.

However, without this assumption, we may work as follows:

If $AB = DE$, then we are done by SAS. So suppose that $AB \neq DE$ and, without loss of generality, that $AB > DE$. Then take G on $\overline{AB}$, such that $GB = DE$. Now by SAS, we get $\triangle GBC \cong \triangle DEF$, whence $\widehat{BGC} \cong \widehat{D} \cong \widehat{A}$, which contradicts the exterior angle theorem. ∎

Problem 3 *Let* l *be a line and* P *a point not on* l. *Construct a line that contains* P *and that is perpendicular to* l.

Solution: Use a compass to take points A, B on l such that $PA \cong PB$. Then use the compass to construct a point P' on the opposite side of $\overline{AB}$ from P, such that $AP' \cong BP'$. We claim that $PP' \perp l$.

Let Q be the point of intersection of PP' with l. By construction $AP \cong BP$, whence $\widehat{PAQ} \cong \widehat{PBQ}$. Also by construction $AP \cong BP, AP' \cong BP'$ and $PP' \cong PP'$, whence by SSS, $\triangle PAP' \cong \triangle PBP'$, and, therefore $\widehat{APQ} \cong \widehat{BPQ}$. Therefore $\widehat{AQP} \cong 180^o - \widehat{PAQ} - \widehat{APQ} \cong 180^o - \widehat{PBQ} - \widehat{BPQ} \cong \widehat{BQP}$. But these two angles are also supplementary angles, whence $PP' \perp l$. ∎

Problem 4 *Let* l *be a line and* P *a point not on* l. *Construct a line that contains* P *and that meets* l *at a* 45^o *angle. Construct a line that contains* P *and that meets* l *at a* 30^o *angle. (To do this exercise you need to use the fact that the sum of the angles in any triangle is* 180^o.)

Solution: Use the construction of the previous exercise to find $PQ \perp l$. Then take A on l, such that $AQ \cong PQ$. Obviously, $\widehat{PAQ} \cong \widehat{QPA}$ and these two are also complementary angles, whence $\widehat{PAQ} = 45^o$.

Similarly, take instead of $QA \cong PQ$, $PA \cong 2PQ$. Then $\sin \widehat{PAQ} = \frac{PQ}{PA} = \frac{1}{2}$, whence $\widehat{PAQ} = 30^o$. ∎

Problem 5 *Given triangles $\triangle ABC$ and $\triangle DEF$ such that $\overline{AB} \cong \overline{DE}, \overline{AC} \cong \overline{DF}$ and $\widehat{A} > \widehat{D}$, prove that $BC > EF$.*

Solution: Let $\overline{AG}$ be such that $\widehat{BAG} \cong \widehat{D}$ and $AG \cong DF$. Then, by SAS, $\triangle ABG \cong \triangle DEF$, whence $AG \cong DF \cong AC$. This yields $\widehat{AGC} \cong \widehat{ACG}$. We now have:

$$\widehat{BGC} \cong \widehat{BGA} + \widehat{AGC} \cong \widehat{EFD} + \widehat{ACG} \cong \widehat{BGA} + \widehat{BCG} + \widehat{ACB} > \widehat{BCG}.$$

Therefore $BC > BG \cong EF$. ■

Problem 6 *Given BC and $A_1, \ldots, A_n$, prove that $BC \leq BA_1 + A_1A_2 + \ldots + A_nC$.*

Solution: By induction on n. For $n = 1$, we get $BC \leq BA_1 + A_1C$,by the triangle inequality. Suppose that the given inequality holds for $n = k$, i.e., that $BC \leq BA_1 + A_1A_2 + \ldots + A_kC$. Consider $k + 1$ points $A_1, \ldots, A_{k+1}$. We then have

$$\begin{array}{rcl} BC & \leq & BA_1 + A_1A_2 + \ldots + A_{k-1}A_{k+1} + A_{k+1}C \\ & \leq & BA_1 + A_1A_2 + \ldots + (A_{k-1}A_k + A_kA_{k+1}) + A_{k+1}C \\ & = & BA_1 + A_1A_2 + \ldots + A_{k-1}A_k + A_kA_{k+1} + A_{k+1}C \end{array}$$

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Problem 7 *Prove that the sum of the angles in a convex n-sided figure is $(n-2)180^o$.*

Solution: Again by induction on the number of sides n. For $n = 3$, the result reduces to the sum of the angles in a convex triangle being 180^o. Suppose the result is true for $n = k$. For $n = k+1$, we get

$$\begin{array}{rcl} \widehat{A_1} + \widehat{A_2} + \ldots + \widehat{A_{k+1}} & = & \widehat{A_1} + \widehat{A_2} + \ldots + \widehat{A_{k-2}} + A_{k-2}\widehat{A_{k-1}}A_{k+1} + A_{k-1}\widehat{A_{k+1}}A_1 + \\ & & A_{k+1}\widehat{A_{k-1}}A_k + \widehat{A_k} + A_k\widehat{A_{k+1}}A_{k-1} \\ & = & (k-2)180^o + 180^o \\ & = & (k-1)180^o \\ & = & ((k+1)-2)180^o. \end{array}$$

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Problem 8 *Prove that a parallelogram $ABCD$ is a rectangle if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ are congruent.*

Solution: First suppose that $ABCD$ is a rectangle. Compare the right triangles $\triangle ABC$ and $\triangle ADB$. They have $AD \cong BC, AB \cong AB, \widehat{A} \cong \widehat{B}$. Thus, by SAS, we get $\triangle ABC \cong \triangle BAD$ and, therefore $AC \cong BD$.

Suppose conversely, that $AC \cong BD$. Then, by SSS, $\triangle ABC \cong \triangle BAD$, whence $\widehat{BAD} \cong \widehat{ABC}$. But these two are supplementary angles, whence each is a right angle and $ABCD$ is a rectangle. ■

Problem 9 *Prove that a parallelogram $ABCD$ is a rhombus if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular to each other.*

Solution: First suppose that $ABCD$ is a rhombus. We have $AB \cong AD$, whence $\widehat{ABD} \cong \widehat{ADB}$. Also, by SSS $\triangle ABC \cong \triangle ADC$, whence $\widehat{BAC} \cong \widehat{CAD}$. These two angle equalities yield $\widehat{BPA} \cong \widehat{DPA}$, where P is the point of intersection of the two diagonals. But these two are also supplementary angles, whence they are both right angles and the diagonals are perpendicular.

Suppose conversely, that $\overline{AC} \perp \overline{BD}$. We have by ASA, $\triangle ABP \cong \triangle CDP$, whence $AP = CP$ and, thus, by SAS, $\triangle ABP \cong \triangle CBP$. This yields $AB = BC$. ■