EXAM 1: SOLUTIONS - MATH 351
INSTRUCTOR: George Voutsadakis

Problem 1 1. *Show that* $3^{2n} - 1$ *is exactly divisible by 8 for* $n \geq 1$.

2. *Prove that* $n(n-1)2^{n-2} = \sum_{k=1}^{n-1} k(n-k)\binom{n}{k}$.

Solution:

1. We work by induction on n. For $n = 1$, $3^{2\cdot 1} - 1 = 8$ is divisible by 8. Suppose that $3^{2n} - 1$ is divisible by 8. We then have

$$\begin{aligned} 3^{2(n+1)} - 1 &= 3^{2n+2} - 1 \\ &= 3^2 3^{2n} - 1 \\ &= 9 \cdot 3^{2n} - 9 + 9 - 1 \\ &= 9(3^{2n} - 1) + 8. \end{aligned}$$

The first summand is divisible by 8 by the induction hypothesis whereas the second summand is obviously divisible by 8. Hence $3^{2(n+1)} - 1$ is also divisible by 8.

2. By the Binomial Theorem we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

If we take partial derivatives of both sides with respect to x we get

$$n(x+y)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1} y^{n-k}.$$

Now, if we take partial derivatives with respect to y, we get

$$n(n-1)(x+y)^{n-2} = \sum_{k=1}^{n-1} k(n-k)\binom{n}{k} x^{k-1} y^{n-k-1},$$

whence, by substituting $x = y = 1$, we get

$$n(n-1)2^{n-2} = \sum_{k=1}^{n-1} k(n-k)\binom{n}{k}.$$

■

Problem 2 1. *Give a* formal *definition of the terms* **subgraph**, **induced subgraph** *and* **spanning subgraph**. *Then explain how many induced and how many spanning subgraphs a labelled graph* G *has.*

2. *Find a 3-regular graph M such that the graph H below is an induced subgraph of M. Prove that it is impossible to find such an M that has just one vertex more than H has.*

Solution:

1. Let $G = \langle V, E\rangle$ be a graph. $H = \langle V', E'\rangle$ is a *subgraph* of G if G' is a graph, $V' \subseteq V$ and $E' \subseteq E$. V' is an *induced subgraph* if $E' = E \cap \mathcal{P}_2(V')$. Finally, it is a *spanning subgraph* if $V' = V$.

   A labelled graph $G = \langle V, E\rangle$ has $2^{|V|}$ induced subgraphs, one for each subset of its vertex set, and $2^{|E|}$ spanning subgraphs, one for each subset of its edge set.

2. Let $M = \langle V, E\rangle$, such that
$$V = \{a, b, c, d, e, f\}$$
   and
$$E = \{\{a, b\}, \{a, c\}, \{b, c\}, \{a, d\}, \{d, e\}, \{d, f\}, \{e, f\}, \{b, e\}, \{c, f\}\}.$$
   It is obvious that H is an induced subgraph of M. M could not possibly have only one additional vertex because that vertex would have degree 3 whereas H lacks at least four edges, one incident with each of b, c and two incident with d. ■

Problem 3 1. *Define* formally *the notion of* **isomorphism** *between graphs and list at least 4 isomorphism invariants.*

2. *Draw two different 3-regular graphs on six vertices. Prove that they are not isomorphic.*

Solution:

1. Let $G = \langle V, E\rangle, G' = \langle V', E'\rangle$ be two graphs. G and G' are *isomorphic*, written $G \cong G'$, if there exists a bijection $f : V \to V'$, such that $\{v_1, v_2\} \in E$ if and only if $\{f(v_1), f(v_2)\} \in E'$, for all $v_1, v_2 \in V$. The following are isomorphism invariants:

   - The number of vertices in a graph.
   - The number of edges in a graph.
   - The degree sequence in a graph.
   - The distance between two vertices of unique degrees.

Please, note that there are many other isomorphism invariants that could have been used here. Your book have a more complete list.

2. Let $G = \langle V, E\rangle$ and $G' = \langle V, E'\rangle$, where $V = \{a, b, c, d, e, f\}$ and

$$E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, e\}, \{b, f\}, \{c, e\}, \{c, f\}, \{d, e\}, \{d, f\}\},$$

$$E' = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, e\}, \{c, f\}, \{d, e\}, \{d, f\}, \{e, f\}\}.$$

Then G and G' are 3-regular on 6 vertices and they are not isomorphic because, for instance, G does not have any subgraph isomorphic to C_3 whereas G' does. ■

Problem 4 *1. Define* formally *the operations of* **join** *and of* **cartesian product** *of graphs. What familiar graph is $K_m + K_n$ isomorphic to? Explain informally.*

2. Show that, given a positive integer n, there exists a self-complementary graph G with $|V(G)| = 4n$.

Solution:

1. Let $G = \langle V, E\rangle$ and $G' = \langle V', E'\rangle$ be two graphs. The *join* $G+G'$ is the graph that has as vertex set the disjoint union of V and V' and as edge set the set that consists of the disjoint union of E and E' together with all edges of the form $\{v, v'\}, v \in V, v' \in V'$.

   Now assume that $V = \{v_1, v_2, \dots, v_m\}$ and $V' = \{u_1, u_2, \dots, u_n\}$. The *cartesian product* $G \times G'$ has as vertex set the set $\{(i, j) : 1 \le i \le m, 1 \le j \le n\}$ and as set of edges the collection of all pairs $\{(i, j), (k, l)\}, 1 \le i, k \le m, 1 \le j, l \le n$, such that

   - $i = k$ and $\{u_j, u_l\} \in E'$,
   - $j = l$ and $\{v_i, v_k\} \in E$.

   $K_m + K_n \cong K_{m+n}$, since all edges between vertices in K_m are inherited from K_m, all edges between vertices in K_n are inherited from K_n and all edges between a vertex of K_m and a vertex of K_n are inserted by the definition of a join.

2. The following figure may help you do the proof on your own:

$$\begin{tikzcd} K_n \arrow[r, dashed, no head] & \overline{K_n} \arrow[d, dashed, no head] \\ K_n \arrow[r, dashed, no head] & \overline{K_n} \end{tikzcd}$$

   The dashed lines suggest that all edges between the corresponding components exist in the graph. ■

Problem 5 *1. Give the* formal *definitions of a* **tree**, **spanning tree** *and* **minimum spanning tree**.

2. *Use Kruskal's algorithm to find the minimum spanning tree of the following weighted graph. Depict all iterations of the algorithm* clearly*!!*

Solution:

1. A *tree* is a connected acyclic graph. A *spanning tree* of a graph G is a spanning subgraph of G that is a tree. A *minimum spanning tree* of a weighted graph G is a spanning tree of minimum weight among all spanning trees of G.

2. Note that this can be done in many different ways depending on which list will be the initial ordered list of edges that the algorithm will start with. Our list here is shown below

edge	cg	gi	ac	bd	fg	dh	eh	ad	cf	eg	fi	hi	ab	cd	be
weight	1	1	2	2	2	2	2	3	3	3	3	3	4	5	7

Kruskal's algorithm inserts all first 8 edges and stops because the graph has 9 vertices. The weight of the minimum spanning tree found is 15. ■