HOMEWORK 2 - MATH 351
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Problem 1 *Explain why it is impossible for the list of degrees of the vertices of a graph to be (a)* $5,4,2,2,2,1,1$ *(b)* $10,6,3,2,2,1,1,1$

Solution: (a) The sum of the degrees is an odd number.
(b) Since there are only 8 vertices, the maximum degree of any vertex may not exceed 7. ■

Problem 2 *Draw three different* 2*-regular graphs.*

Solution: We have the 3-cycle $C_3 = \langle V_1, E_1 \rangle$, with

$$V_1 = \{0,1,2\}, E_1 = \{\{0,1\},\{1,2\},\{0,2\}\}.$$

Another one is the 4-cycle $C_4 = \langle V_2, E_2 \rangle$, with

$$E_2 = \{0,1,2,3\}, V_2 = \{\{0,1\},\{1,2\},\{2,3\},\{0,3\}\}.$$

Finally, the union $C_3 \cup C_3 = \langle V_3, E_3 \rangle$, with

$$V_3 = \{0,1,2,3,4,5\}, E_3 = \{\{0,1\},\{1,2\},\{0,2\},\{3,4\},\{4,5\},\{3,5\}\}$$

is a disconnected 2-regular graph. ■

Problem 3 *Draw a* 3*-regular graph on six vertices.*

Solution: $K_4 = \langle V_1, E_1 \rangle$, with

$$V_1 = \{0,1,2,3\}, E_1 = \{\{0,1\},\{0,2\},\{0,3\},\{1,2\},\{1,3\},\{2,3\}\}.$$

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Problem 4 *Explain why there is no* 3*-regular graph on seven vertices.*

Solution: The sum of the degrees of such a graph would have been $7 \cdot 3 = 21$, which is an odd number, contradicting the fact that it has to be twice the number of edges in the graph. ■

Problem 5 *Describe all* 2*-regular graphs (connected and disconnected).*

Solution: A graph is a connected 2-regular graph iff it is an n-cycle for some n. A graph is a disconnected 2-regular graph iff it is the disjoint union of cycles. ■

Problem 6 *Find a formula in terms of m and n for the number of edges in the complete bipartite graph $K_{m,n}$.*

Solution: Let e be the number of edges. Then, since m vertices have degree n and the remaining m vertices have degree n, we get $2e = nm + mn$, i.e., $2e = 2mn$ or $e = mn$. ■

Problem 7 *Draw a 3-regular disconnected graph on eight vertices.*

Solution: The graph $K_4 \cup K_4$ fills the bill. This is the graph $K_4 \cup K_4 = \langle V_1, E_1 \rangle$, with

$$V_1 = \{0, 1, 2, 3, 4, 5, 6, 7\},$$

$$E_1 = \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{4, 7\}, \{5, 6\}, \{5, 7\}, \{6, 7\}\}.$$

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Problem 8 *(a) Determine which of the graphs below are subgraphs of graph G. Explain.*
(b) Which of the graphs below are induced subgraphs of graph G? Explain.

Solution: H is a subgraph of G, since all edges in H are also edges in G. J is not a subgraph of G since $\{b, d\} \in E(J)$ but $\{b, d\} \notin E(G)$. Finally, K is a subgraph of G.

H is an induced subgraph since every edge not in H has at least an adjacent vertex not in H. J is not even a subgraph, let alone an induced subgraph, and, finally, K is a subgraph but not an induced subgraph, since $a, e \in V(K)$, but $\{a, e\} \notin E(K)$, despite the fact that $\{a, e\} \in E(G)$. ■

Problem 9 *Why can there be no $a - f$ path of length six in graph G below?*

Solution: Since a path is not allowed to visit a single vertex twice and the graph G has only 6 vertices, any path between two of its vertices may have length at most 5. ■

Problem 10 *Consider the disconnected graph on n vertices consisting of two components: K_{n-1} and K_1. How many edges does it have? Show that this is the maximum number of edges a disconnected graph on n vertices can have.*

Solution: The number of edges in K_1 is zero, whence the number of edges in $K_{n-1} \cup K_1$ is equal to the number of edges in K_{n-1}, which is $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$.

Any other disconnected graph will have two or more components. If it has more than two components, then the number of its edges may be increased by adding an edge to join two of its components. So the number of edges of a disconnected graph with more than two components is always smaller than the number of edges of some graph with only two components. Furthermore, it is clear that among all disconnected graphs with two components of orders k and $n-k$, the ones with the largest size are the ones where the two components are K_k and K_{n-k}. So to prove the assertion of the problem, it suffices to show that the number of edges in $K_k \cup K_{n-k}$ is less than or equal to $\frac{(n-1)(n-2)}{2}$.

$$\begin{aligned}
\frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} &= \frac{k^2-k+n^2-nk-n-nk+k^2+k}{2} \\
&= \frac{2k^2+n^2-2nk-n}{2} \\
&= \frac{n(n-1)-2k(n-k)}{2} \\
&\leq \frac{n(n-1)-2(n-1)}{2} \\
&= \frac{(n-1)(n-2)}{2}.
\end{aligned}$$

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Problem 11 *Three nonisomorphic graphs have degree sequences* 3, 2, 2, 1, 1, 1*. Construct them.*

Solution: $G_1 = \langle V_1, E_1 \rangle$, with

$$V_1 = \{0,1,2,3,4,5\}, E_1 = \{\{0,1\},\{0,2\},\{0,3\},\{1,4\},\{4,5\}\}.$$

$G_2 = \langle V_2, E_2 \rangle$, with

$$V_2 = \{0,1,2,3,4,5\}, E_2 = \{\{0,1\},\{0,2\},\{0,3\},\{1,4\},\{2,5\}\}.$$

$G_3 = \langle V_3, E_3 \rangle$, with

$$V_3 = \{0,1,2,3,4,5\}, E_3 = \{\{0,1\},\{0,2\},\{0,3\},\{1,2\},\{4,5\}\}.$$

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Problem 12 *Three nonisomorphic graphs have degree sequence* 5, 3, 2, 2, 1, 1, 1, 1*. Construct them.*

Solution: $G_1 = \langle V_1, E_1 \rangle$, with

$$V_1 = \{0,1,2,3,4,5,6,7\}, E_1 = \{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{1,2\},\{1,6\},\{6,7\}\}.$$

$G_2 = \langle V_2, E_2 \rangle$, with

$$V_2 = \{0,1,2,3,4,5,6,7\}, E_2 = \{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{1,6\},\{2,6\},\{6,7\}\}.$$

$G_3 = \langle V_3, E_3 \rangle$, with

$$V_3 = \{0,1,2,3,4,5,6,7\}, E_3 = \{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{1,2\},\{2,3\},\{6,7\}\}.$$

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Problem 13 *There are three nonisomorphic graphs besides P_7 that have degree sequence* 2, 2, 2, 2, 2, 1, 1*. Draw them.*

Solution: $G_1 = \langle V_1, E_1 \rangle$, with

$$V_1 = \{0,1,2,3,4,5,6\}, E_1 = \{\{0,1\},\{1,2\},\{0,2\},\{3,4\},\{4,5\},\{5,6\}\}.$$

$G_2 = \langle V_2, E_2 \rangle$, with

$$V_2 = \{0,1,2,3,4,5,6\}, E_2 = \{\{0,1\},\{1,2\},\{2,3\},\{0,3\},\{4,5\},\{5,6\}\}.$$

$G_3 = \langle V_3, E_3 \rangle$, with

$$V_3 = \{0,1,2,3,4,5,6\}, E_3 = \{\{0,1\},\{1,2\},\{2,3\},\{3,4\},\{0,4\},\{5,6\}\}.$$

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Problem 14 *Show that the sequence $n-1, 3, 3, 3, \ldots, 3$ of length $n \geq 4$ is graphical.*

Solution: This is the degree sequence of the wheel $W_{1,n-1}$. ■

Problem 15 *Given a graph G with degree sequence $d_1, d_2, d_3, \ldots, d_k, d_{k+1}, \ldots, d_n$ show that there exists a graph H with degree sequence (out of order perhaps) $k, d_1+1, d_2+1, d_3+1, \ldots, d_k+1, d_{k+1}, \ldots, d_n$ by showing how to construct H from G.*

Solution: H is obtained from G by adding one extra vertex to G and edges connecting this new vertex to the k vertices of G that have in G degrees $d_1, \ldots, d_k$. ■

Problem 16 *(a) Prove that C_4 and $K_{2,2}$ are isomorphic.*
(b) Prove that $K_{1,2}$ and P_3 are isomorphic.

Solution: (a) Suppose $C_4 = \langle V_1, E_1 \rangle$, with

$$V_1 = \{0,1,2,3\}, E_1 = \{\{0,1\},\{1,2\},\{2,3\},\{0,3\}\}$$

and $K_{2,2} = \langle V_2, E_2 \rangle$, with

$$V_2 = \{a,b,c,d\}, E_2 = \{\{a,c\},\{a,d\},\{b,c\},\{b,d\}\}.$$

Then the mapping $0 \mapsto a, 1 \mapsto c, 2 \mapsto b, 3 \mapsto d$ is an isomorphism of C_4 with $K_{2,2}$.
(b) Let $K_{1,2} = \langle V_1, E_1 \rangle$, with

$$V_1 = \{a,b,c\}, E_1 = \{\{a,b\},\{a,c\}\}$$

and $P_3 = \langle V_2, E_2 \rangle$, with

$$V_2 = \{0,1,2\}, E_2 = \{\{0,1\},\{1,2\}\}.$$

Then $a \mapsto 1, b \mapsto 0$ and $c \mapsto 2$ is an isomorphism between $K_{1,2}$ and P_3. ■

Problem 17 *Draw two nonisomorphic disconnected subgraphs of C_5 that have four vertices.*

Solution: Let $C_5 = \langle V, E \rangle$, with

$$V = \{0, 1, 2, 3, 4\}, E = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{0, 4\}\}.$$

The first subgraph is $G_1 = \langle V_1, E_1 \rangle$, with

$$V_1 = \{0, 1, 2, 3\}, E_1 = \{\{0, 1\}, \{1, 2\}\}.$$

The second subgraph is $G_2 = \langle V_2, E_2 \rangle$, with

$$V_2 = \{0, 1, 2, 3\}, E_2 = \{\{0, 1\}, \{2, 3\}\}.$$

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Problem 18 *Explain why the graphs G and H below are* not *isomorphic by finding some characteristic that distinguishes them from one another.*

Solution: Note that the geodesic between the two vertices of degree 3 in G has length 2, whereas the two vertices of degree 3 in H are adjacent vertices. ■

Problem 19 *Draw the wheel $W_{1,6}$.*

Solution:

■

Problem 20 *How many edges does $W_{1,n}$ have?*

Solution: It has $2n$ edges, n of them on the "cycle" and another n connecting the "cycle" with the "central" vertex. ■

Problem 21 *Prove that Q_3 is isomorphic to the mesh $M(2, 2, 2)$.*

Solution: Let $Q_3 = \langle V_1, E_1 \rangle$, with

$$V_1 = \{0, 1, 2, 3, 4, 5, 6, 7\},$$

$$E_1 = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{0, 3\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{4, 7\}, \{0, 4\}, \{1, 5\}, \{2, 6\}, \{3, 7\}\}$$

and $M(2,2,2) = \langle V_2, E_2 \rangle$, with

$$V_2 = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\},$$

$$E_2 = \{\{(0,0,0),(0,0,1)\},\{(0,0,1),(0,1,1)\},\{(0,1,1),(0,1,0)\},\{(0,0,0),(0,1,0)\},$$

$$\{(1,0,0),(1,0,1)\},\{(1,0,1),(1,1,1)\},\{(1,1,1),(1,1,0)\},\{(1,0,0),(1,1,0)\},$$

$$\{(0,0,0),(1,0,0)\},\{(0,0,1),(1,0,1)\},\{(0,1,1),(1,1,1)\},\{(0,1,0),(1,1,0)\}\}.$$

Then, the mapping defined by

$$0 \mapsto (0,0,0), 1 \mapsto (0,0,1), 2 \mapsto (0,1,1), 3 \mapsto (0,1,0),$$

$$4 \mapsto (1,0,0), 5 \mapsto (1,0,1), 6 \mapsto (1,1,1), 7 \mapsto (1,1,0)$$

is an isomorphism between Q_3 and $M(2,2,2)$. ■

Problem 22 *Consider K_3 with vertices a, b and c. Now obtain a new graph, A, by adding two more vertices r and s and edges ra and sb. Show that A is self-complementary.*

Solution: We have $A = \langle V, E \rangle$, with

$$V = \{a,b,c,r,s\}, E = \{\{a,b\},\{bc\},\{a,c\},\{a,r\},\{b,s\}\}.$$

Therefore $\overline{A} = \langle V, \overline{E} \rangle$, where

$$\overline{E} = \{\{c,r\},\{c,s\},\{r,s\},\{r,b\},\{s,a\}\}.$$

Hence A is isomorphic to $\overline{A}$ via the isomorphism

$$a \mapsto s, b \mapsto r, c \mapsto c, r \mapsto a, s \mapsto b.$$

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Problem 23 *Assume that G and H are graphs where $V(G) = \{u_1, \ldots, u_m\}$ and $V(H) = \{v_1, \ldots, v_n\}$. Let (i,j) be a vertex in $G \times H$. Prove that* $\deg(i,j) = \deg(u_i) + \deg(v_j)$.

Solution: The neighbors (h,k) of (i,j) in $G \times H$ are divided into two disjoint sets: those for which $h = i$ and v_k is a neighbor of v_j and those for which $k = j$ and u_h is a neighbor of u_i. The first set has $\deg(v_j)$ elements and the second set has $\deg(u_i)$ elements. Therefore

$$\deg(i,j) = \deg(u_i) + \deg(v_j).$$

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Problem 24 *Show that $K_{m,n} \cong \overline{K_m} + \overline{K_n}$.*

Solution: Let $K_{m,n} = \langle V, E\rangle$, with

$$V = \{1, 2, \dots, m-1, m, m+1, \dots, m+n\},$$

$$E = \{\{1, m+1\}, \dots, \{1, m+n\}, \dots, \{m, m+1\}, \dots, \{m, m+n\}\},$$

$\overline{K_m} = \langle V_1, \emptyset\rangle$, with $V_1 = \{1, 2, \dots, m\}$, and $\overline{K_n} = \langle V_2, \emptyset\rangle$, with $V_2 = \{1', 2', \dots, n'\}$. Then $\overline{K_m}+\overline{K_n} = \langle V_1 \cup V_2, V_1 \times V_2\rangle$, whence the mapping

$$1 \mapsto 1, \dots, m \mapsto m, m+1 \mapsto 1', \dots, m+n \mapsto n'$$

is an isomorphism between $K_{m,n}$ and $\overline{K_m} + \overline{K_n}$. ■

Problem 25 *Explain why the complement of $G + H$ is disconnected for all pairs of graphs G and H.*

Solution: Let $V(G)$ be the vertex set of G and $V(H)$ the vertex set of H. Every edge connecting a vertex in $V(G)$ with a vertex in $V(H)$ is an edge in $G+H$. Therefore no edge in $\overline{G+H}$ connects a vertex in $V(G)$ with a vertex in $V(H)$ in $\overline{G+H}$. Therefore $\overline{G+H}$ has at least two connected components. ■

Problem 26 *If $G \cong H$, show that $\overline{G} \cong \overline{H}$.*

Solution: Let $h : V(G) \to V(H)$ be an isomorphism from G to H. Therefore $\{v_1, v_2\} \in E(G)$ if and only if $\{h(v_1), h(v_2)\} \in E(H)$. Thus we have

$$\begin{aligned} \{v_1, v_2\} \in E(\overline{G}) \quad &\text{iff} \quad \{v_1, v_2\} \notin E(G) \\ &\text{iff} \quad \{h(v_1), h(v_2)\} \notin E(H) \\ &\text{iff} \quad \{h(v_1), h(v_2)\} \in E(\overline{H}), \end{aligned}$$

whence $h : \overline{G} \to \overline{H}$ is also an isomorphism from $\overline{G}$ to $\overline{H}$. ■