HOMEWORK 8: SOLUTIONS - MATH 111
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Problem 1 *Solve the system of equations using the* **Gauss-Jordan** *method* $\left\{\begin{array}{rcl} y & = & x-1 \\ y & = & 6+z \\ z & = & -1-x \end{array}\right\}$

Solution:
The given system can be rewritten as

$$\left\{\begin{array}{rrrrrrr} x & - & y & & & = & 1 \\ & & y & - & z & = & 6 \\ x & & & + & z & = & -1 \end{array}\right\}$$

Thus, using the augmented matrix and the Gauss-Jordan gives

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 6 \\ 1 & 0 & 1 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 6 \\ 0 & 1 & 1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 7 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 2 & -8 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 7 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right].$$

Thus $(x,y,z)=(3,2,-4)$. ■

Problem 2 *Let* $A=\left[\begin{array}{rrrr} -1 & 3 & -5 & 0 \\ 2 & 6 & -1 & 1 \end{array}\right]$ *and* $B=\left[\begin{array}{rr} 1 & 0 \\ 3 & 2 \\ -1 & 7 \\ 1 & -5 \end{array}\right]$. *Compute* $A\cdot B$ *and* $B\cdot A$.

Solution:

$$A{\cdot}B=\left[\begin{array}{rrrr} -1 & 3 & -5 & 0 \\ 2 & 6 & -1 & 1 \end{array}\right]\cdot\left[\begin{array}{rr} 1 & 0 \\ 3 & 2 \\ -1 & 7 \\ 1 & -5 \end{array}\right]=\left[\begin{array}{cc} -1+9+5 & 6-35 \\ 2+18+1+1 & 12-7-5 \end{array}\right]=\left[\begin{array}{rr} 13 & -29 \\ 22 & 0 \end{array}\right].$$

$$B\cdot A=\left[\begin{array}{rr} 1 & 0 \\ 3 & 2 \\ -1 & 7 \\ 1 & -5 \end{array}\right]\cdot\left[\begin{array}{rrrr} -1 & 3 & -5 & 0 \\ 2 & 6 & -1 & 1 \end{array}\right]=\left[\begin{array}{rrrr} -1 & 3 & -5 & 0 \\ 1 & 21 & -17 & 2 \\ 15 & 39 & -2 & 7 \\ -11 & -27 & 0 & -5 \end{array}\right].$$ ■

Problem 3 *Let* $X=\left[\begin{array}{cc} x & 0 \\ 0 & y \end{array}\right]$. *Solve the matrix equation* $X^2=2X+\left[\begin{array}{rr} -1 & 0 \\ 0 & 3 \end{array}\right]$.

Solution:

$X^2 = 2X + \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ implies $\begin{bmatrix} x^2 & 0 \\ 0 & y^2 \end{bmatrix} = \begin{bmatrix} 2x-1 & 0 \\ 0 & 2y+3 \end{bmatrix}$. This, in turn, yields the system of equations $\left\{ \begin{array}{rcl} x^2 & = & 2x-1 \\ y^2 & = & 2y+3 \end{array} \right\}$, i.e., $\left\{ \begin{array}{rcl} x^2-2x+1 & = & 0 \\ y^2-2y-3 & = & 0 \end{array} \right\}$, whence

$$\left\{ \begin{array}{rcl} (x-1)^2 & = & 0 \\ (y-3)(y+1) & = & 0 \end{array} \right\},$$

which gives $x = 1$ and $y = -1$ or $y = 3$. Thus, the two solutions for X are $X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $X = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. ∎

Problem 4 *Compute the inverses of the matrices* $A = \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} -2 & -1 \\ 6 & -3 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{rr|rr} -1 & -2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 1 & 2 & -1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{rr|rr} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 1 & 0 & -3 & -2 \\ 0 & 1 & 1 & 1 \end{array}\right].$$

Thus $A^{-1} = \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix}$. Similarly,

$$\left[\begin{array}{rr|rr} -2 & -1 & 1 & 0 \\ 6 & -3 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 6 & -3 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -6 & 3 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{rr|rr} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{6} \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 1 & 0 & -\frac{1}{4} & \frac{1}{12} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{6} \end{array}\right].$$

Thus $B^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{12} \\ -\frac{1}{2} & -\frac{1}{6} \end{bmatrix}$. ∎

Problem 5 *Find the inverse of the matrix* $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -2 & -1 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right].$$

Thus $A^{-1} = \left[\begin{array}{ccc} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right].$ ■

Problem 6 *Let A, B be sets in a universe U. Suppose that $n(A) = 5, n(B) = 9, n(A \cap B) = 2$ and $n(U) = 20$. What is $n((A \cup B)^c)$?*

Solution:

We have $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 5 + 9 - 2 = 12$. Hence $n((A \cup B)^c) = n(U) - n(A \cup B) = 20 - 12 = 8$. ■

Problem 7 *Suppose that A, B and C are sets in a universe U. If $n(A) = 22, n(B) = 20, n(C) = 12, n(A \cap B) = 11, n(A \cap C) = 9, n(B \cap C) = 5, n(A \cap B \cap C) = 3$ and $n(U) = 44$, fill in the number of elements of each region in the appropriate Venn diagram.*

Solution:

We have

region	number
$A \cap B \cap C$	3
$A \cap B \cap C^c$	8
$A \cap B^c \cap C^c$	5
$A \cap B^c \cap C$	6
$A^c \cap B \cap C$	2
$A^c \cap B \cap C^c$	7
$A^c \cap B^c \cap C^c$	12
$A^c \cap B^c \cap C$	1

■

Problem 8 *At a pow-wow in Arizona, Native Americans from all over the Southwest came to participate in the ceremonies. A coordinator of the pow-wow took a survey and found that 15 families brought food, costumes and crafts, 25 families brought food and crafts, 42 families brought food, 20 families brought costumes and food, 6 families brought costumes and crafts but no food, 4 families brought crafts, but neither food nor costumes, 10 families brought none of the three items and 18 families brought costumes but not crafts. (a) How many families were surveyed? (b) How many families brought costumes? (c) How many families brought food or costumes?*

Solution:

Let F denote the subset consisting of those families that brought food, C the subset of those that brought costumes and R the subset of those that brought crafts. Then the number of elements in each area of the Venn diagram is shown in the following table

region	number
$F \cap C \cap R$	15
$F \cap C \cap R^c$	5
$F \cap C^c \cap R^c$	12
$F \cap C^c \cap R$	10
$F^c \cap C \cap R$	6
$F^c \cap C \cap R^c$	13
$F^c \cap C^c \cap R^c$	10
$F^c \cap C^c \cap R$	4

Thus 75 families were surveyed, 39 of which brought costumes and 61 of which brought food or costumes. ■