# FINAL EXAM: SOLUTIONS - MATH 325
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**Problem 1** *(a) State Ceva's Theorem.*

*(b) Give a proof of Ceva's Theorem.*

**Solution:**

(a) If three Cevians $AX, BY, CZ$, one through each vertex of a triangle $ABC$, are concurrent, then
$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = 1.$$

(b) Let $P$ be the common point of the three Cevians. We have
$$\frac{BX}{XC} = \frac{(ABX)}{(AXC)} = \frac{(PBX)}{PXC)} = \frac{(ABX)-(PBX)}{(AXC)-(PXC)} = \frac{(ABP)}{(CAP)}.$$
Similarly, $\frac{CY}{YA} = \frac{(BCP)}{(ABP)}$ and $\frac{AZ}{ZB} = \frac{(CAP)}{(BCP)}$. Therefore
$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = \frac{(ABP)}{(CAP)}\frac{(BCP)}{(ABP)}\frac{(CAP)}{(BCP)} = 1.$$

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**Problem 2** *(a) Give the definition of the Euler line.*

*(b) Prove the Theorem of Gergonne: The lines joining the vertices of a triangle with the points where the incircle touches the opposite sides are concurrent.*

**Solution:**

(a) The orthocenter, centroid and circumcenter of any triangle are collinear. The line on which these three points lie is called the *Euler line* of the triangle.

(b) Let $X, Y, Z$ be the points of tangency of the incircle with the sides $BC, AC, AB$, respectively. Since $AY$ and $AZ$ are the tangents to the incircle from $A$, we have $AY = AZ = x$. Similarly, $BX = BZ = y$ and $CX = CY = z$. Therefore, we have
$$\frac{AZ}{ZB}\frac{BX}{CX}\frac{CY}{AY} = \frac{x}{y}\frac{y}{z}\frac{z}{x} = 1.$$
Thus, by the converse to Ceva's Theorem, we conclude that $AX, BY$ and $CZ$ are concurrent. ■

**Problem 3** *(a) Define the 9 point circle.*

(b) *Prove that the three feet of the altitudes of a triangle and one of the midpoint of one of its sides are cocyclic.*

**Solution:**

(a) The feet of the three altitudes of any triangle, the midpoints of the three sides and the midpoints of the segments from the three vertices to the orthocenter, all lie on the same circle, called the *nine-point circle* of the triangle.

(b) First note that the median of a right triangle corresponding to the right angle divides the triangle into two isosceles triangles. This will be used in the string of equalities below.

Now let $AD, BE, CF$ be the three altitudes, $AM$ the median and $H$ the orthocenter of a triangle $ABC$. We have

$$\begin{aligned}
\widehat{FEM} &= 180 - \widehat{AEF} - \widehat{MEC} \\
&= 180 - \widehat{FHA} - \hat{C} \\
&= 180 - \widehat{DHC} - \hat{C} \\
&= \widehat{DHE} - \widehat{DHC} \\
&= \widehat{CHE} \\
&= \widehat{FHB} \\
&= \widehat{FDB} \\
&= 180 - \widehat{FDM}.
\end{aligned}$$

Hence $D, E, F$ and $M$ are cocyclic. ■

**Problem 4** *(a) Define the power of a point with respect to a circle.*

(b) *Show that the power of a given point lying outside a given circle with respect to that circle is equal to the square of the length of one of the tangents from the point to the circle.*

**Solution:**

(a) For any circle of radius $R$ and any point $P$ distant $d$ from the center, we call $d^2 - R^2$ the *power* of $P$ with respect to the circle.

(b) Let $P$ be the point outside the circle with center $O$ and radius $R$ and let $PT$ be one of the two tangents to the circle through $P$. Then $OT \perp PT$, whence, by the Pythagorean Theorem, $PO^2 = PT^2 + OT^2$, i.e., $PT^2 = PO^2 - R^2$, which is, by definition, the power of $P$ with respect to the circle. ■

**Problem 5** *(a) State Simson's Theorem.*

(b) *Give a proof of Simson's Theorem.*

**Solution:**

(a) The feet of the perpendiculars from a point to the sides of a triangle are collinear if and only if the point lies on the circumference.

(b) Let $P$ be a point lying on the arc $AC$ of the circumcircle and denote by $A_1, B_1, C_1$, respectively, the feet of the perpendiculars to $BC, AC, AB$ from $P$. We then have $\widehat{APC} = 180 - \hat{B} = \widehat{C_1PA_1}$, whence, by subtracting $\widehat{APA_1}$, we get $\widehat{A_1PC} = \widehat{C_1PA}$. But $A_1, C, P, B_1$ are cocyclic, whence $\widehat{A_1PC} = \widehat{A_1B_1C}$, and $A, B_1, P, C_1$ are cocyclic, whence $\widehat{C_1PA} = \widehat{C_1B_1A}$. Therefore $\widehat{A_1B_1C} = \widehat{C_1B_1A}$. Hence $A_1, B_1$ and $C_1$ are collinear.

The steps above may be reversed to show the converse. ■

**Problem 6** *(a) State Pierre Varignon's Theorem.*

*(b) Give a proof of Varignon's Theorem.*

**Solution:**

(a) The figure formed when the midpoints of the sides of a quadrangle are joined in order is a parallelogram, and its area is half the area of the quadrangle.

(b) Given the quadrangle $ABCD$, let the midpoints of the sides $AB, BC, CD$ and $DA$ be $P, Q, R$ and $S$, respectively. Then $PS \parallel= QR \parallel= \frac{1}{2}BD$. Thus $PQRS$ is a parallelogram. For the area we have

$$\begin{aligned}(PQRS) &= (ABCD) - (PBQ) - (RDS) - (QCR) - (SAP)\\ &= (ABCD) - \tfrac{1}{4}(ABC) - \tfrac{1}{4}(CDA) - \tfrac{1}{4}(BCD) - \tfrac{1}{4}(DAB)\\ &= (ABCD) - \tfrac{1}{4}(ABCD) - \tfrac{1}{4}(ABCD)\\ &= \tfrac{1}{2}(ABCD).\end{aligned}$$

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**Problem 7** *(a) State Menelaus's Theorem.*

*(b) Give a proof of Menelaus's Theorem.*

**Solution:**

(a) If points $X, Y, Z$ on the sides $BC, CA, AB$ (suitably extended) of $ABC$ are collinear, then $\frac{BX}{CX}\frac{CY}{AY}\frac{AZ}{BZ} = 1$. Conversely, if this equation holds for points $X, Y, Z$ on the three sides, then these three points are collinear.

(b) Let $h_1, h_2, h_3$ be the lengths of the perpendiculars from $A, B, C$ to the line $XY$. Then $\frac{BX}{CX}\frac{CY}{AY}\frac{AZ}{BZ} = \frac{h_2}{h_3}\frac{h_3}{h_1}\frac{h_1}{h_2} = 1$.

Conversely, if $X, Y, Z$ occur on the three sides in such a way that $\frac{BX}{CX}\frac{CY}{AY}\frac{AZ}{BZ} = 1$, let the lines $AB$ and $XY$ meet at $Z'$. Then by the first part, $\frac{BX}{CX}\frac{CY}{AY}\frac{AZ'}{BZ'} = 1$. Hence $\frac{AZ'}{BZ'} = \frac{AZ}{BZ'}$, whence $Z' \equiv Z$ and $X, Y, Z$ are collinear. ■

**Problem 8** *(a) State Pappus's Theorem.*

*(b) If $A, C, E$ are three points on one line, $B, D, F$ on another, and if the two lines $AB$ and $CD$ are parallel to $DE$ and $FA$, respectively, then $EF$ is parallel to $BC$.*

**Solution:**

(a) If $A, C, E$ are three points on one line, $B, D, F$ are three points on another line, and if the three lines $AB, CD, EF$ meet $DE, FA, BC$, respectively, then the three points of intersection $L, M, N$ are collinear.

(b) If $AC \parallel BD$, then we have $ABDE$ and $CDFA$ parallelograms, whence $BD = AE$ and $DF = CA$, whence $BF = CE$. Hence $EFBC$ is also a parallelogram and $EF \parallel BC$.

On the other hand, if $O$ is the point of intersection of $AC$ and $BD$, then $AB \parallel ED$ implies $\frac{OA}{OB} = \frac{OE}{OD}$, whence $OAOD = OEOB$. But $AF \parallel CD$ gives $\frac{OA}{OF} = \frac{OC}{OD}$, whence $OAOD = OCOF$. Therefore $OEOB = OCOF$ which gives $\frac{OC}{OB} = \frac{OE}{OF}$ and, therefore $BC \parallel EF$. ■

**Problem 9** *(a) State Desargues's Theorem.*

*(b) State Pascal's Theorem.*

**Solution:**

(a) If two triangles are perspective from a point and if their pairs of corresponding sides meet, then the three points of intersection are collinear.

(b) If all six vertices of a hexagon lie on a circle and the three pairs of opposite sides intersect, then the three points of intersection are collinear. ■

**Problem 10** *(a) State Brianchon's Theorem.*

*(b) Give the formulas of Brahmagupta and of Heron for the area of a cyclic quadrangle and of a triangle respectively.*

**Solution:**

(a) If all six sides of a hexagon touch a circle, the three diagonals are concurrent (or possibly parallel).

(b) For a cyclic quadrangle with sides $a, b, c, d$, semiperimeter $s$ and area $K$, we have

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

For a triangle with sides $a, b, c$, semiperimeter $s$ and area $K$, we have

$$K = \sqrt{s(s-a)(s-b)(s-c)}.$$

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