HOMEWORK 4: SOLUTIONS - MATH 325
INSTRUCTOR: George Voutsadakis

Problem 1 *Two circles are in contact internally at a point T. Let the chord AB of the larger circle be tangent to the smaller circle at a point P. Then the line TP bisects $A\hat{T}B$.*

Solution:

Let C be the point of intersection of AB with the tangent to the two circles at T. Then we have

$$\begin{aligned}\widehat{BTP} &= \widehat{PTC} - \widehat{BTC} \\ &= \widehat{TPC} - \widehat{BAT} \\ &= \widehat{BAT} + \widehat{PTA} - \widehat{BAT} \\ &= \widehat{PTA}.\end{aligned}$$

■

Problem 2 *The points where the extended altitudes meet the circumcircle form a triangle similar to the orthic triangle.*

Solution:

Let AA' be the altitude and A'' the pont where it intersects the circumference. Similarly for B', B'' and C' and C''. Then we have

$$\begin{aligned}\widehat{A''CA'} &= \widehat{BAA''} \\ &= \widehat{BB'A'} \\ &= \widehat{A'CH},\end{aligned}$$

where H is the orthocenter. Therefore, the two triangles HCA' and $A''CA'$ are congruent, which shows that $HA' = A'A''$. By symmetry $HB' = B'B''$ and $HC' = C'C''$. Thus

$$\frac{A'B'}{A''B''} = \frac{B'C'}{B''C''} = \frac{A'C'}{A''C''} = \frac{1}{2}.$$

This entails the required similarity. ■

Problem 3

1. *What point on the circle has CA as its Simson line?*
2. *Are there any points that lie on their own Simson lines? What lines are these?*

Solution:

(a) Let P be such a point. For CA to be its Simson line, the pedal points of P to AB and to BC must be A and C, respectively, because they must lie on AC as well. Therefore $PA \perp BA$ and $PC \perp BC$. Thus $\widehat{BAP} = \widehat{BCP} = 90^o$, which shows that P is the antidiametric point of B on the circumference.

(b) The three vertices of ABC lie on their own Simson lines, which are the altitudes of the triangle.

■

Problem 4 *The tangents at two points B and C on a circle meet at A. Let $A_1B_1C_1$ be the pedal triangle of the isosceles triangle ABC for an arbitrary point P on the circle, as in Figure 2.5B, page 41. Then $PA_1^2 = PB_1 \times PC_1$.*

Solution:
We have

$$\begin{aligned} \widehat{PC_1A_1} &= \widehat{PBA_1} \\ &= \widehat{PCA} \\ &= \widehat{PA_1B_1}. \end{aligned}$$

Similarly, $\widehat{PB_1A_1} = \widehat{PA_1C_1}$. These two equalities establish the similarity of PA_1C_1 and PB_1A_1. Hence we get

$$\frac{PA_1}{PB_1} = \frac{PC_1}{PA_1}, \text{ i.e., } PA_1^2 = PB_1 \cdot PC_1.$$

■

Problem 5 *If a point P lies on the arc CD of the circumcircle of a square $ABCD$, then $PA(PA+PC) = PB(PB+PD)$.*

Solution:
By the Pythagorean Theorem, we have that $BD^2 = AB^2 + AD^2 = 2AB^2$, whence $BD = \sqrt{2}AB$. Therefore, applying Ptolemy's Theorem to the quadrilaterals $ABPD$ and $ABCP$, we get

$$AB \cdot PD + PB \cdot AD = PA \cdot BD, \text{ i.e., } PB + PD = \sqrt{2}PA.$$

Similarly

$$AB \cdot PC + PA \cdot BC = PB \cdot AC, \text{ i.e., } PA + PC = \sqrt{2}PB.$$

These two now yield

$$\begin{aligned} PA(PA+PC) &= PA \cdot \sqrt{2}PB \\ &= PB \cdot \sqrt{2}PA \\ &= PB(PB+PD). \end{aligned}$$

■

Problem 6 *If a circle cuts two sides and a diagonal of a parallelogram $ABCD$ at points P, R, Q as shown in Figure 2.6A, page 43, then $AP \times AB + AR \times AD = AQ \times AC$. (*Hint*: Apply Theorem 2.61 to the quadrilateral $PQRA$ and then replace the sides of PQR by the corresponding sides of the similar triangle CBA.)*

Solution:

Applying Ptolemy's Theorem to $APQR$ we obtain

$$AP \cdot AB + AR \cdot AD = AQ \cdot AC.$$

Now observe that $\widehat{RPQ} = \widehat{RAQ} = \widehat{ACB}$ and $\widehat{PRQ} = \widehat{PAQ} = \widehat{BAC}$. Therefore QPR is similar to BCA, whence

$$\frac{RQ}{AB} = \frac{PQ}{BC} = \frac{RP}{AC}.$$

This now, combined with the previous equation, yields

$$AP \cdot AB + AR \cdot BC = AQ \cdot AC,$$

whence

$$AP \cdot AB + AR \cdot AD = AQ \cdot AC.$$

■