HOMEWORK 7: SOLUTIONS - MATH 325
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Problem 1 *If A, C, E are three points on one line, B, D, F on another, and if the two lines AB and CD are parallel to DE and FA, respectively, then EF is parallel to BC.*

Solution:

If $AC \parallel BD$, then we have $ABDE$ and $CDFA$ parallelograms, whence $BD = AE$ and $DF = CA$, whence $BF = CE$. Hence $EFBC$ is also a parallelogram and $EF \parallel BC$.

On the other hand, if O is the point of intersection of AC and BD, then $AB \parallel ED$ implies $\frac{OA}{OB} = \frac{OE}{OD}$, whence $OAOD = OEOB$. But $AF \parallel CD$ gives $\frac{OA}{OF} = \frac{OC}{OD}$, whence $OAOD = OCOF$. Therefore $OEOB = OCOF$ which gives $\frac{OC}{OB} = \frac{OE}{OF}$ and, therefore $BC \parallel EF$. ∎

Problem 2 *Let C and F be any points on the respective sides AE and BD of a parallelogram $AEBD$. Let M and N denote the points of intersection of CD and FA and of EF and BC. Let the line MN meet DA at P and EB at Q. Then $AP = QB$.*

Solution:

Note that by Pappus's Theorem, the points M and N are collinear with the center O of the given parallelogram. Thus $OPA \cong OQB$ and, therefore, $AP = QB$. ∎

Problem 3 *If two triangles are perspective from a point, and two pairs of corresponding sides are parallel, the two remaining sides are parallel.*

Solution:

Suppose ABC and $A'B'C'$ are perspective from the point O and that $AB \parallel A'B'$ and $AC \parallel A'C'$. Then $\frac{OB}{OB'} = \frac{OA}{OA'} = \frac{OC}{OC'}$, whence $BC \parallel B'C'$. ∎

Problem 4 *If a hexagon $ABCDEF$ has two opposite sides BC and EF parallel to the diagonal AD, and two opposite sides CD and FA parallel to the diagonal BE, while the remaining sides DE and AB also are parallel, then the third diagonal CF is parallel to AB, and the centroids of ACE and BDF coincide.*

Solution:

Let AB and CD meet at V, CD and FE meet at W and AB and FE meet at U. Then $UADE$ and $AFWD$ are parallelograms, whence $UE = AD = FW$, and, therefore, $UF = EW = BC$, where the last equality follows from the fact that $BCWE$ is also a parallelogram. Thus, $BCUF$ is also a parallelogram and, hence $CF \parallel AB$.

Now let X, Y be the points where BE meet CF and AD, respectively. Then $CDEX$ and $BCDY$ are parallelograms, and their centers A' and F' being the midpoints of DX and DB lie on a line parallel to BX and AF. Since $AF = BX = 2F'A'$, the lines AA' and FF' meet at a point G, such that $AG = 2GA'$ and $FG = 2GF'$. But AA' and FF' are medians of ACE and BDF, whence the two triangles share G as their common centroid. ∎

Problem 5 *1. If five of the six vertices of a hexagon lie on a circle, and the three pairs of opposite sides meet at three collinear points, then the sixth vertex lies on the same circle.*

2. For a cyclic quadrangle $ABCE$ with no parallel sides, the tangents at A and C meet on the line joining $AB \cdot CE$ and $BC \cdot EA$.

Solution:

1. Let A, B, C, D and E lie on a circle and let F' be the point where the side AF meets that circle. The points L, M, N, of intersection of AB and ED, CD and AF, BC and EF, are collinear by hypothesis. But the point F' lies on the line EN by Pascal's Theorem. Since F, F' are points of intersection of EN and AF, they must coincide.

2. Apply Pascal's Theorem to the degenerate cyclic hexagon $ABCCEA$.

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Problem 6 *In Figure 3.9D, page 79, the line PQ joining the other two points of contact also passes through the intersection of the diagonals.*

Solution:
the conclusion follows from Brianchon's Theorem by taking the degenerate hexagon $BQCEPF$ and noting that BE, QP and CF are its diagonals. ■