# EXAM 3 - MATH 341

Thursday, April 3, 2003
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Read each problem very carefully before starting to solve it. Each question is worth 10 points. It is necessary to show your work. Correct answers without explanations are worth 0 points.

GOOD LUCK!!

1. (a) Let $H = \{\sigma \in S_4 : \sigma(2) = 2\}$. Show that $H$ is a subgroup of $S_4$ and find its order.
   (b) Show that if $\sigma \in S_n$ and $|\sigma| = 2$, then $\sigma$ is a product of disjoint 2-cycles.
2. (a) Find all the cosets of $\langle 6 \rangle$ in $\mathbf{Z}_{12}$ and all the cosets of $\langle 6 \rangle$ in the subgroup $\langle 2 \rangle$ of $\mathbf{Z}_{12}$.
   (b) Let $H$ be a subgroup of a group $G$. Show that for any $a \in G$ we have $|Ha| = |H|$.
3. Let $H$ be a subgroup of a group $G$. Show that the map $a \mapsto a^{-1}$ determines a one-one, onto map between the left cosets of $H$ and the right cosets of $H$.
4. (a) Let $\phi : G \to G'$ be a homomorphism, $K = \mathrm{Kern}(\phi)$ and $a \in G$. Show that $\{x \in G : \phi(x) = \phi(a)\} = aK$, the left coset of $K$ to which the element $a$ belongs.
   (b) Show that $U(14) \cong U(18)$.
5. (a) For $r \in \mathbb{R}^*$ let $rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$. Show that $H = \{rI : r \in \mathbb{R}^*\}$ is a normal subgroup of $\mathrm{GL}(2, \mathbb{R})$.
   (b) Let $Z(G)$ be the center of a group $G$. Show that if the index $[G : Z(G)] = p$, a prime, then $G$ is Abelian.