HOMEWORK 3: SOLUTIONS - MATH 490
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Problem 1 *Let $(X, \mathcal{T})$ be a topological space that is metrizable. Prove that for each pair a, b of distinct points of X, there are open sets O_a and O_b containing a and b respectively, such that $O_a \cap O_b = \emptyset$. Prove that the topological space of Example 7 on page 72 is not metrizable.*

Solution: Let $\langle X, d\rangle$ be the metric space whose open sets form the collection $\mathcal{T}$. Consider $a, b \in X$. The two open balls $O_a = B(a; \frac{d(a,b)}{2})$ and $O_b = B(b; \frac{d(a,b)}{2})$ are in $\mathcal{T}$ and $a \in O_a, b \in O_b$, with $A_a \cap O_b = \emptyset$.

The space is $\langle \mathbf{N}^*, \mathcal{T}\rangle$ where $\mathcal{T} = \{\emptyset, O_1, O_2, \ldots\}$, with $O_n = \{n, n+1, \ldots\}, n \geq 1$. It is clearly not metrizable since given $m, n \in \mathbf{N}^*$, with $m < n$, any open set that contains m has to also contain n. Hence, there are no disjoint open neighborhoods of m, n. ■

Problem 2 *Let $(X, \mathcal{T})$ be a topological space. Prove that $\emptyset, X$ are closed sets, that a finite union of closed sets is a closed set, and that an arbitrary intersection of closed sets is a closed set.*

Solution:

We have $C(\emptyset) = X$ and $C(X) = \emptyset$ and, since both X and $\emptyset$ are open, we have that $\emptyset$ and X are both closed as complements of open sets.

Now given $F_1, \ldots, F_n$ closed, we have that

$$C(F_1 \cup \ldots \cup F_n) = C(F_1) \cap \ldots \cap C(F_n),$$

which is a finite intersection of open sets and is, therefore, open. Thus $F_1 \cup \ldots \cup F_n$ is closed.

Finally, given an arbitrary collection $\{F_\alpha\}_{\alpha \in I}$ of closed sets, we get

$$C(\bigcap_{\alpha \in I} F_\alpha) = \bigcup_{\alpha \in I} C(F_\alpha)$$

which is an arbitrary union of open sets and is, therefore, open. hence $\bigcap_{\alpha \in I} F_\alpha$ is closed. ■

Problem 3 *Prove that in a discrete topological space, each subset is simultaneously open and closed.*

Solution:

Recall that in a discrete space every subset is open. Therefore every subset is open and, at the same time, has an open complement. Therefore every subset is simultaneously open and closed. ■

Problem 4 *A family $\{A_\alpha\}_{\alpha\in I}$ of subsets is said to be* mutually disjoint *if for each distinct pair β, γ of indices $A_\beta \cap A_\gamma = \emptyset$. Prove that for each subset A of a topological space $(X, \mathcal{T})$, the three sets $\mathrm{Int}(A), \mathrm{Bdry}(A)$ and $\mathrm{Int}(C(A))$ are mutually disjoint and that $X = \mathrm{Int}(A) \cup \mathrm{Bdry}(A) \cup \mathrm{Int}(C(A))$.*

Solution:

Recall that $x \in \mathrm{Int}(A)$ if there exists an open set O, such that $x \in O \subseteq A$. Thus, this open set contains x and does not intersect $C(A)$, which shows that $x \notin \overline{C(A)}$, whence $x \notin \mathrm{Bdry}(A)$. Conversely, if $x \in \mathrm{Bdry}(A)$, then $x \in \overline{C(A)}$, which shows that there does not exist open set O containing x such that $O \subseteq A$. Hence $x \notin \mathrm{Int}(A)$. This shows that $\mathrm{Int}(A) \cap \mathrm{Bdry}(A) = \emptyset$. The proofs of the other two mutual disjointness relations are similar.

Now suppose that $x \notin \mathrm{Bdry}(A)$. Hence $x \notin \overline{A}$ or $x \notin \overline{C(A)}$. If the first condition holds, then there exists an open set O containing x and such that $O \cap A = \emptyset$. Thus $x \in O \subseteq C(A)$. This shows that $x \in \mathrm{Int}(C(A))$. In the second case, one shows, similarly, that $x \in \mathrm{Int}(A)$. Hence $X = \mathrm{Int}(A) \cup \mathrm{Bdry}(A) \cup \mathrm{Int}(C(A))$. ■

Problem 5 *In the real line prove that the boundary of the open interval (a, b) and the boundary of the closed interval $[a, b]$ is $\{a, b\}$.*

Solution:

We have that $\overline{(a,b)} = [a, b]$ and $\overline{C((a, b))} = \overline{(-\infty, a] \cup [b, \infty)} = (-\infty, a] \cup [b, \infty)$. Therefore $\mathrm{Bdry}((a, b)) = \overline{(a, b)} \cap \overline{C((a, b))} = \{a, b\}$. One handles the closed interval similarly.

■

Problem 6 *Let A be a subset of a topological space. Prove that $\mathrm{Bdry}(A) = \emptyset$ if and only if A is open and closed.*

Solution:

Suppose, first, that A is both open and closed. Then $C(A)$ is also both open and closed. Therefore $\overline{A} = A$ and $\overline{C(A)} = C(A)$. Therefore $\mathrm{Bdry}(A) = \overline{A} \cap \overline{C(A)} = A \cap C(A) = \emptyset$.

Suppose, conversely, that $\mathrm{Bdry}(A) = \emptyset$. Therefore $\overline{A} \cap \overline{C(A)} = \emptyset$. We show that both A and $C(A)$ are open. Suppose that $a \in A$. Then $a \in \overline{A}$, whence, since $\overline{A} \cap \overline{C(A)} = \emptyset$, $a \notin \overline{C(A)}$. Thus, there exists an open set O, such that $a \in O$ and $O \cap C(A) = \emptyset$. But then $a \in O \subseteq A$, whence A is a neighborhood of a. Since a was arbitrary, A is a neighborhood of each of its points and is, therefore, open. A very similar argument, with the roles of A and $C(A)$ interchanged, shows that $C(A)$ is also open. ■

Problem 7 *A subset A of a topological space $(X, \mathcal{T})$ is said to be* dense in X *if $\bar{A} = X$. Prove that if for each open set O we have $A \cap O \neq \emptyset$, then A is dense in X.*

Solution:

We need to show that, if, for all open sets O, $A \cap O \neq \emptyset$, then $\overline{A} = X$. To this aim, let $x \in X$ and O open with $x \in O$. But, by the hypothesis, $O \cap A \neq \emptyset$, whence, since O was arbitrary, $x \in \overline{A}$. But x was also arbitrary, whence $X \subseteq \overline{A}$. The reverse inclusion is obvious, and, therefore, $\overline{A} = X$ and A is dense in X. ■

Problem 8 *Let a function $f : X \to Y$ be given. Prove that $f : (X, 2^X) \to (Y, \mathcal{T}')$ is always continuous, as is $f : (X, \mathcal{T}) \to (Y, \{\emptyset, Y\})$, where $\mathcal{T}'$ is any topology on Y and $\mathcal{T}$ is any topology on X.*

Solution:

We show first, that, for an arbitrary topology $\mathcal{T}'$ on Y, the function $f : (X, 2^X) \to (Y, \mathcal{T}')$ is continuous. Suppose $O \in \mathcal{T}'$. Then $f^{-1}(O) \subseteq X$, whence $f^{-1}(O) \in 2^X$ and, therefore, $f^{-1}(O)$ is open in $(X, 2^X)$. This proves that f is continuous.

Now let $\mathcal{T}$ be an arbitrary topology on X and consider $f : (X, \mathcal{T}) \to (Y, \{\emptyset, Y\})$. Let O be open in $(Y, \{\emptyset, Y\})$. Then $O = \emptyset$ or $O = Y$. therefore $f^{-1}(O) = \emptyset$ or $f^{-1}(O) = X$. In either case $f^{-1}(O) \in \mathcal{T}$, whence $f^{-1}(O)$ is open in X and f is continuous. ■