# HOMEWORK 6: SOLUTIONS - MATH 490
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**Problem 1** *Prove that the product of two locally connected topological spaces is locally connected.*

**Solution:**

Let $a = (a_1, a_2) \in X_1 \times X_2$ and consider a neighborhood $N$ of $a$ in $X \times Y$. Then $p_1(N)$ is a neighborhood of $a_1$ in $X_1$ and $p_2(N)$ is a neighborhood of $a_2$ in $X_2$. Hence, by the local connectedness of $X_1, X_2$, there exists a connected neighborhood $U_1$ of $a_1$ in $p_1(N)$ and a connected neighborhood $U_2$ of $a_2$ in $p_2(N)$. It is not hard to see that then $U_1 \times U_2$ is a connected neighborhood of $a$ in $N$. Thus $X_1 \times X_2$ is locally connected at $a$ and, since $a$ was arbitrary, $X_1 \times X_2$ is locally connected. ■

**Problem 2** *Verify that in a topological space $X$*

1. *if there is a path with initial point $A$ and terminal point $B$, then there is a path with initial point $B$ and terminal point $A$, and*
2. *if there is a path connecting points $A$ and $B$ and a path connecting points $B$ and $C$, then there is a path connecting points $A$ and $C$.*

**Solution:**

1. Suppose that $f : I \to X$ is a path in $X$, such that $f(0) = A$ and $f(1) = B$. Then $f' : I \to X$, defined by $f'(t) = f(1-t)$ is a path in $X$ with $f'(0) = f(1) = B$ and $f'(1) = f(0) = A$.
2. Suppose that $f : I \to X, g : I \to X$ are paths in $X$ with $f(0) = A, f(1) = B$ and $g(0) = B, g(1) = C$. Define $h : I \to X$, by
$$h(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$
Then $h$ is a path in $X$, such that $h(0) = f(0) = A$ and $h(1) = g(1) = C$.

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**Problem 3** *If $A$ and $B$ are path-connected subsets of a topological space $X$ and $A \cap B \neq \emptyset$, then $A \cup B$ is path-connected.*

**Solution:**

Let $x, y \in A \cup B$. If both $x, y \in A$ or both $x, y \in B$, then, there is a path connecting $x$ and $y$ in $A \cup B$ by the path-connectedness of $A$ or $B$, respectively. So, suppose that $x \in A$ and $y \in B$ (the case $x \in B$ and $y \in A$ may be handled similarly). Then, since $A \cap B \neq \emptyset$, there exists $z \in A \cap B$. By the path-connectedness of $A$, there exists a path $f : I \to A$

such that $f(0) = x$ and $f(1) = z$ and, by the path-connectedness of $B$, there exists a path $g : I \to B$, such that $g(0) = z$ and $g(1) = y$. Now take the path $h$ in $A \cup B$, as defined in the previous problem. This is a path from $x$ to $y$ in $A \cup B$, whence $A \cup B$ is path-connected as was to be shown. ■

**Problem 4** *Let $X, Y$ be topological spaces and $f : X \to Y$ be a continuous function with $f(x) = y$. Let $g, g'$ be closed paths at $x \in X$. Prove that $fg \cong fg'$ whenever $g \cong g'$.*

**Solution:**

Let $H : I \times I \to X$ be a homotopy from $g$ to $g'$. That is,

$$H(s,t) = \begin{cases} x, & \text{if } s = 0 \text{ or } 1 \\ g(s), & \text{if } t = 0 \\ g'(s), & \text{if } t = 1 \end{cases}$$

Define a homotopy $H_f : I \times I \to Y$ from $fg$ to $fg'$ as follows:

$$H_f(s,t) = f(H(s,t)), \quad \text{for all } s, t \in I.$$

Then $H_f(0,t) = f(H(0,t)) = f(x) = y$ and, similarly, $H_f(1,t) = y$. Also $H_f(s,0) = f(H(s,0)) = f(g(s)) = (fg)(s)$ and, similarly, $H_f(s,1) = (fg')(s)$. ■

**Problem 5** *Two groups $G$ and $G'$ are called* isomorphic *if there are homomorphisms $h : G \to G'$ and $h' : G' \to G$ such that $h'h$ is the identity mapping on $G$ and $hh'$ is the identity mapping on $G'$. Prove that if $f : X \to Y$ is a homeomorphism of the topological space $X$ with the space $Y$ such that $f(x) = y$, then $\Pi(X, x)$ is isomorphic to $\Pi(Y, y)$.*

**Solution:**

Let $F : \Pi(X, x) \to \Pi(Y, y)$ be defined by $F([g]) = [fg]$, for all $[g] \in \Pi(X, x)$. That this mapping is well defined, is shown by the previous problem. It is not difficult to check that it satisfies the homomorphism property:

$$\begin{aligned} F([g] \cdot [g']) &= F([g \cdot g']) \\ &= [f(g \cdot g')] \\ &= [fg \cdot fg'] \\ &= [fg] \cdot [fg'] \\ &= F([g]) \cdot F([g']) \end{aligned}$$

Finally, $F$ is a bijection because the mapping $F^{-1} : \Pi(Y, y) \to \Pi(X, x)$, defined by

$$F^{-1}([h]) = [f^{-1}h], \quad \text{for all } [h] \in \Pi(Y, y),$$

is an inverse of $F$. We omit the details of this verification. ■

**Problem 6** *An isomorphism of a group $G$ with itself is called an* automorphism. *Let $f$ and $f'$ be paths in a space $Z$ with $f(0) = f'(1) = z$ and $f(1) = f'(0) = y$. Let $f' \cdot f^{-1}$ be the path defined by*

$$(f' \cdot f^{-1})(t) = \begin{cases} f'(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ f^{-1}(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

*Prove that $a_{f'}a_f$ is an automorphism of $\Pi(Z, y)$ such that $a_{f'}a_f([g]) = [f' \cdot f^{-1}] \cdot [g] \cdot [f' \cdot f^{-1}]^{-1}$.*

**Solution:**
We need to show that $a_{f'}a_f$ is a bijective homomorphism. ■

**Problem 7**

1. *Prove that the real line $\mathbb{R}$ is not compact.*
2. *Prove that every finite subset of a topological space is compact.*

**Solution:**

1. Consider the collection of open sets $\mathcal{O} = \{O_n\}_{n \in \mathbf{N}^*}$, such that $O_n = (-n, n)$. Clearly, $\mathcal{O}$ is an open covering of $\mathbb{R}$. This open covering has no finite subcovering. Therefore $\mathbb{R}$ is not compact.
2. Let $A = \{a_1, \ldots, a_n\}$ be a finite subset of $X$. Consider any open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of $A$. Then, since $\mathcal{U}$ is a covering of $A$, there exist $\alpha_1, \ldots, \alpha_n \in I$, such that $a_i \in U_{\alpha_i}, i = 1, 2, \ldots, n$. Therefore $A = \{a_1, \ldots, a_n\} \subseteq \cup_{i=1}^n U_{\alpha_i}$ and $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ is a finite subcovering of $A$ by elements of $\mathcal{U}$. Thus $A$ is compact.

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**Problem 8** *Let $X$ be a topological space. A family $\{F_\alpha\}_{\alpha \in I}$ of subsets of $X$ is said to have the* finite intersection property *if for each finite subset $J$ of $I$, $\cap_{\alpha \in J} F_\alpha \neq \emptyset$. Prove that $X$ is compact if and only if for each family $\{F_\alpha\}_{\alpha \in I}$ of closed subsets of $X$ that has the finite intersection property, we have $\cap_{\alpha \in I} F_\alpha \neq \emptyset$.*

**Solution:**
By Theorem 2.8, $X$ is compact if and only whenever a family $\{F_\alpha\}_{\alpha \in I}$ of closed sets is such that $\cap_{\alpha \in I} F_\alpha = \emptyset$, then there exists a finite subset of indices $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, such that $\cap_{i=1}^n F_{\alpha_i} = \emptyset$. Thus, by taking the contrapositive, $X$ is compact if and only if for each family $\{F_\alpha\}_{\alpha \in I}$ of closed subsets of $X$ that has the finite intersection property, we have $\cap_{\alpha \in I} F_\alpha \neq \emptyset$. ■