A Study of the Basics of Fourier Transform

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In mathematics, a transform changes one image to another through some process such as rotation, reflection, translation, or dilations. In Fourier transforms, a function f(t) that may consist of many parts that are continuous or discrete, is transformed to be represented by a single expression. Fourier transforms have many real world applications such as the chemistry application in spectroscopy. In spectroscopy, f(t) is taken from the intensity of the light beam at the output point and this function is a function of optical path difference.[2] This equation represents an ideal situation which is not practical for lab experiments so a more realistic equation is needed to properly interpret the data. In the process of nuclear magnetic resonance spectroscopy, the rotation of a molecule's spin around a magnetic field is being recorded. The function f(t) is formed from interpreting the time at which it takes the spin to return to rest forming a sinusoidal graph. Through Fourier transform, it takes this function f(t) and produces the spectrum. In spectroscopy, it must be taken into account that because the bounds are not infinite, a function f(δ) must be consider where T is an upper limit[2].

$$f(\delta) = \begin{cases} 1\; if\; 0 \leq \delta \leq T \\ 0\; if\; \delta > T \end{cases}$$

Taking this discrete interpretation of a continuous Fourier Transform can produce a spectrum with real world applications. In spectroscopy, the data goes from time to frequency. This is done by introducing a wave number which can be used to take the time a process takes to show the frequency of a certain occurrence in the data.

The Fourier Transform equation can be represented in different ways, the interpretation remains the same, the only difference comes from the fact that the inverse transform equation must be changed to fit with the original equation[1]. The Fourier Transform is indicated by F(ω).

1) $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt$

This equation is how the Fourier Transform is found when f(t) is given which is often the case in real world applications. Inversely, if F(ω) is given, we can use the inverse transform to solve for f(t)[1].

2) $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega$

Proof of Inverse Transform:

First, the identity given must be proven as well as an important form of the identity.

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)dt = \phi(0)$$

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t}\, dt$$

$$= e^{-i\omega\,(0)}$$

$$= e^{0}$$

$$F(\omega) = 1$$

The proof of this inversion formula uses the following Identity equations (Equations 3 and 4)[1].

3) $\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,d\omega$

4) $\phi(t) = \int_{-\infty}^{\infty} \phi(x)\delta(t-x)dx$

Using equation 1, a substitution for F(ω) is made and also replacing t with x to obtain equation 5.

5) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx\right]e^{i\omega t}\,d\omega$

Separating the integrals into two different parts matching the integration parts of dx and dω equation 6 is formed.

6) $= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)dx \int_{-\infty}^{\infty} e^{i\omega(t-x)}d\omega$

Using substitution for u = (t-x) and using the identity in equation 3, equation 7 is formed.

7) $= \int_{-\infty}^{\infty} f(x)\delta(t-x)dx$

By the identity (equation 4), we have the final equation equal to f(t) which was our desired result and thus shows the proof of the inversion formula.

Fourier Transforms are a group of complex equations that are often times difficult to prove. There are several properties which are used to help simplify some of the more difficult tasks of proving the validity of transforms. In these formulas, the use of a special format to show the Fourier integral and its inverse. This format is g(t) ↔ G(ω). The first of these simple theorems is the Linearity Theorem (Equation 8)[1].

8) $a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)$

The proof of this theorem is shown by inserting the f(t) of equation 8 into equation 1 (equation 9).

9) $G(\omega) = \int_{-\infty}^{\infty}(a_1 f_1(t) + a_2 f_2(t))e^{-i\omega t}\,dt$

Using distribution for the integral and the exponential, equation 10 is formed.

10) $G(\omega) = \int_{-\infty}^{\infty} a_1 f_1(t)e^{-i\omega t}\,dt + \int_{-\infty}^{\infty} a_2 f_2(t)e^{-i\omega t}\,dt$

To form equation 11, the constants can be pulled out in front of the integral because they will not be affected by integration limits.

11) $G(\omega) = a_1\int_{-\infty}^{\infty} f_1(t)e^{-i\omega t}\,dt + a_2\int_{-\infty}^{\infty} f_2(t)e^{-i\omega t}\,dt$

It can now be clearly seen that each part of the equation resembles equation 1 with the addition of a constant out in front. Substitution can be used to insert $F_1(\omega)$ and $F_2(\omega)$ to form equation 12.

12) $G(\omega) = a_1F_1(\omega) + a_2F_2(\omega)$

Equation 12 is our desired result and completes the proof of the linearity theorem. The next theorem is the Symmetry theorem which states that if f(t) ⟷ F(ω), then F(t) ⟷ 2πf(-ω)[1].

The proof of this theorem starts with using substitution to insert 2πf(-ω) into equation 2 to form equation 13.

13) $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi f(-\omega)e^{i\omega t}\, d\omega$

Equation 14 is formed by simply pulling out the constant 2π which ends up cancelling out with the 1/2π.

14) $g(t) = \int_{-\infty}^{\infty} f(-w)e^{i\omega t}\, d\omega$

The next step in this proof uses u substitution where u = -ω (Equation 15).

15) $g(t) = -\int_{\infty}^{-\infty} f(u)e^{-iut}\, du$

To form equation 16, the integration limits are swapped which also removes the negative.

16) $g(t) = \int_{-\infty}^{\infty} f(u)e^{-iut}\, du$

17) g(t) = F(t)

Equation 17 is formed from the use of equation 1 which concludes with our desired result and thus completing the proof of the symmetry theorem[1].

The next simple theorem is the time scaling theorem which states the formula in equation 18.

18) $f(at) \leftrightarrow \frac{1}{|a|}F(\frac{\omega}{a})$ for real values of a

For a > 0 and inserting f(at) into equation 1 then equation 19 is created.

19) $G(\omega) = \int_{-\infty}^{\infty} f(at)e^{-i\omega t}\,dt$

Using substitution for x = at the equation changes to equation 20.

20) $G(\omega) = \frac{1}{a}\int_{-\infty}^{\infty} f(x)e^{-i(\omega/a)x}\,dx$

Using the formula from equation 1 we can substitute to get the result found in equation 21.

21) $G(\omega) = \frac{1}{a}F(\frac{\omega}{a})$

This is our desired result and thus completes the proof of the time scaling formula. The next formula (equation 22) is the time shifting theorem[1].

22) $f(t - t_0) \leftrightarrow F(\omega)e^{-it_0\omega}$

The proof of the time shifting theorem starts by substituting into equation 1 which results in equation 23.

23) $G(\omega) = \int_{-\infty}^{\infty} f(t - t_0)e^{-i\omega t}\,dt$

Substituting in for x = t – t_0 where t = x + t_0 equation 24 results.

24) $G(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega(t_0+x)}dx$

To form equation 25, the constants are removed from the integral.

25) $G(\omega) = e^{-i\omega t_0}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$

Using equation 1, F(ω) is substituted in and equation 26 results.

26) $G(\omega) = F(\omega)e^{-i\omega t_0}$

This is our desired result thus the time shifting theorem has been proved. The next simple theorem is the frequency shifting theorem for which ω_0 is real (equation27).

27) $e^{i\omega t_0} f(t) \leftrightarrow F(\omega - \omega_0)$

Inserting this into equation 1 we get equation 28.

28) $G(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega_0 t} e^{-i\omega t}\, dt$

Equation 29 results from the combining of like terms in equation 28.

29) $G(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i(\omega-\omega_0)t} dt$

Using equation 1 the resulting formula is found (equation 30).

30) $G(\omega) = \ F(\omega - \omega_0)$

This is our desired resulting formula and therefore the frequency shifting theorem has been proven. Another simple theorem used is the time differentiation theorem (equation 31)[1].

31) $\frac{d^n f}{dt^n} \leftrightarrow (i\omega)^n F(\omega)$

The proof of equation 31 can be done using induction in which equation 32 shows the basic case.

32) $n = 0\ \ f(t) \leftrightarrow F(\omega)$

33) $n = 1\ \ f^{'}(t) \leftrightarrow (i\omega) F(\omega)$

Proof of equation 33 starts by using equation 1.

34) $\int_{-\infty}^{\infty} f^{'}(t) e^{-i\omega t}\, dt$

Using integration by parts we obtain equation 35.

35) $f(t)e^{-i\omega t}\,|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-i\omega)e^{-i\omega t}\,dt$

The first term goes to 0 which leaves us with equation 36.

36) $i\omega \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$

Using equation 1 we end up with the result in equation 33 as desired. The next step is to show true for n+1 so that by induction we can prove this theorem[1].

37) $G(\omega) = \int_{-\infty}^{\infty} \frac{d^{n+1}f}{dt^{n+1}} e^{-i\omega t}\,dt$

The next step is to separate the differentials to simplify the process. Equation 38 is the result of this separation.

38) $= \int_{-\infty}^{\infty} \frac{d}{dt}\left[\frac{d^n f}{dt^n}\right] e^{-i\omega t}\,dt$

Using integration by parts we end up with equation 39.

39) $= \frac{d^n f}{dt^n} e^{-i\omega t}\,|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\omega)\frac{d^n f}{dt^n} e^{-i\omega t}\,dt$

We can assume that the first part of the equation goes to 0 and we can pull out the constants that are inside the integral. By equations 31 and 36 as well as the constants, we can obtain equation 40.

40) $= (i\omega)(i\omega)^n F(\omega)$

By combining like terms we end up with equation 41 which is our desired result.

$$= (i\omega)^{n+1} F(\omega)$$

The next simple theorem is the frequency differentiation which is stated in equation 41.

41) $(-it)^n f(t) \leftrightarrow \frac{d^n F(\omega)}{d\omega^n}$

The proof of this can be started using our symmetry theorem (if f(t) ⟷ F(ω), then F(t) ⟷ 2πf(-ω) which when used results in equation 42.

42) $\frac{d^n F(t)}{dt^n} \leftrightarrow 2\pi(-it)^n f(-\omega)$

To obtain equation 43 we use substitution in the inversion formula.

43) $G(t) = \int_{-\infty}^{\infty} (-it)^n f(-\omega) e^{i\omega t}\, d\omega$

Using substitution where u=-ω we obtain equation 44.

44) $= \int_{-\infty}^{\infty} (-it)^n f(u) e^{-iut}\, du$

From equation 41 and taking out the constants we obtain equation 45 which is our desired result.

45) $= \frac{d^n F(t)}{dt^n}$

Another simple theorem used is the conjugate function theorem which states the formula found in equation 46.

46) $\dot{f}(t) \leftrightarrow \dot{F}(-\omega)$

47) $G(\omega) = \int_{-\infty}^{\infty} (f_1 + if_2) e^{-i\omega t}\, dt$

48) $\dot{G}(\omega) = \int_{-\infty}^{\infty} (f_1 - if_2) e^{i\omega t}\, dt$

From equations 38 and 39 we have the resulting equation 40.

49) $\dot{G}(-\omega) = \int_{-\infty}^{\infty} (f_1 - if_2) e^{-i\omega t}\, dt$

By the definition of conjugates and using equation 1 we get the final result (equation 41).

50) $\dot{G}(-\omega) = \int_{-\infty}^{\infty} \dot{f}(t)e^{-i\omega t}\, dt$

This is our desired result and thus ends the proof of the conjugate function theorem. The last simple theorem used is the moment theorem which states equation 51 and m_n is equal to equation 52[1].

51) $(-i)^n m_n = \frac{d^n F(0)}{d\omega^n}$

52) $\int_{-\infty}^{\infty} t^n f(t)dt \quad n = 0,1,2, \ldots$

We start with the base case where n = 0 and ω = 0 (equation 44).

53) $m_0 = F(0)$

Using the MacLaurin Series we obtain equation 54.

54) $G(\omega) = \int_{-\infty}^{\infty} f(t) \left[\sum_{n=0}^{\infty} \frac{(-i\omega t)^n}{n!}\right] dt$

Using equation 52 we obtain equation 55.

55) $G(\omega) = \sum_{n=0}^{\infty} (-i)^n m_n \frac{\omega^n}{n!}$

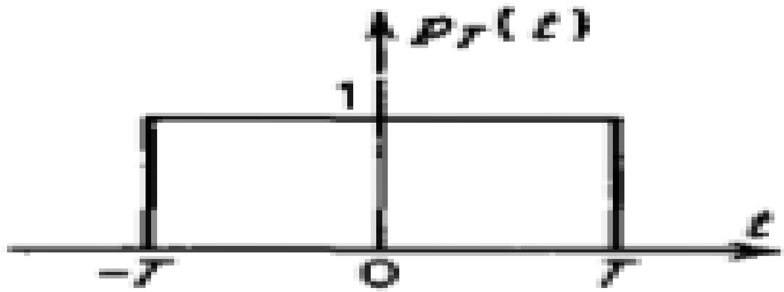
By the Taylor Series we obtain equation 56.

56) $G(\omega) = \sum_{n=0}^{\infty} \frac{d^n F(0)}{d\omega^n} \frac{\omega^n}{n!}$

This concludes in our desired result which ends the proof of the moment theorem.

Examples

There are several uses for the previously discussed simple theorems. We will use a few examples to show some of these uses on important functions. Example 1 is involving the function $p_T(t)$. This function is show in graph 1[1].



Graph 1[1]

To apply this to Fourier Transforms we must first make the statement found in equation 57 which is what we intend to prove.

57) $p_T(t-t_0) \leftrightarrow \frac{2\sin\,\omega T}{\omega} e^{-i\omega\,t_0}$

It is easier to start by proving equation 58.

58) $g(t) = \ p_T(t)$

We must first insert this formula into equation 1 t solve for G(ω). The next step is to realize that this function only exists as something other than 0 between –T and T so we are able to change our integration limits. Also, this function is equal to 1 throughout these limits which results in the right half of equation 59.

59) $\int_{-\infty}^{\infty} p_T(t) e^{-i\omega t}\, dt = \int_{-T}^{T} e^{-i\omega t}\, dt$

To continue on with this proof it is easier to use the identity found in equation 60.

60) $e^{iy} = \cos y + i \sin y$

Using this identity we obtain equation 61.

61) $= \int_{-T}^{T}(cos\omega t - isin\omega t)dt$

Because this function is an even function and the sine function is odd, this results in the sine part going to 0 and we can remove it from the equation (Equation 62)[1].

62) $= \int_{-T}^{T} cos\omega tdt$

Next we can simply solve the integral resulting in equation 63.

63) $= \frac{1}{\omega} sin\omega t|_{-T}^{T}$

Inserting the limits into the equation we obtain equation 64.

64) $= \frac{2 \sin \omega T}{\omega}$

Using the results obtain from this as well as the simple theorem, time shifting theorem (equation 22) we obtain our result in equation 57 as desired.

Example 2 is to prove that from equation 65 we can obtain equation 66.

65) $\frac{d^n \delta(t)}{dt^n} \leftrightarrow (i\omega)^n$

66) $t^n \leftrightarrow 2\pi i^n \frac{d^n \delta(\omega)}{d\omega^n}$

We must first prove the validity of the above statement of equation 65 before we can show its relationship to equation 66.

We start this proof by using the symmetry simple theorem (symmetry equation found in equation 70) and then substituting into the inversion formula (equation 67).

67. $\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi \frac{d^n \delta(-\omega)}{d(-\omega)^n} e^{-i\omega t}\, d\omega$

We can pull out the constants which will cancel the 2π's and because we know that δ(t) ↔ 1 we can simply remove δ(-ω) from the equation. Also, we can remove $(-1)^n$ from the denominator and pull it out front. After doing these steps we obtain equation 68[1].

68. $\int_{-\infty}^{\infty} (-1)^n \frac{d^n}{d\omega^n} e^{-i\omega t}\, d\omega$

We then have, from equations 44 and 45, the resulting in our desired result thus proving equation 65.

69. $(it)^n$

Next, using the symmetry simple theorem on equation 65 once more, we can obtain equation 66 (equation 70).

70. $(it)^n \leftrightarrow 2\pi \frac{d^n \delta(-\omega)}{d(-\omega)^n}$

We can multiply both sides by i^n to obtain equation 71. Doing multiplication with constants is allowed because they do not contain the variable that is affected by the integration limits. Also the δ(-ω) = δ(ω) because the Fourier Transform of both will equal 1.

71. $i^{2n} t^n \leftrightarrow 2\pi i^n \frac{d^n \delta(\omega)}{d(-\omega)^n}$

We can obtain equation 72, by removing $(-1)^n$ from the right hand side and also by realizing that

$i^{2n} = (-1)^n$ and so we can remove the $(-1)^n$ from both sides of the equation and we are left with equation 66 as desired.

Convolution Theorem

The convolution is similar to the simple theorems however its importance is greater. It is also a formula that is used to help to simplify many formulas[1]. This theorem is not matched up with the simple theorems because it is more complex as well as being used more frequently. The convolution theorem states that when we are given two functions $f_1(x)$ and $f_2(x)$ we can form the integral found in equation 72[1].

72. $f(x) = \int_{-\infty}^{\infty} f_1(y) f_2(x-y)dy$

This function f(x) is known as the convolution of these two functions and is typically denoted as equation 73.

73. $f(x) = f_1(x) * f_2(x)$

The time convolution theorem states that if $f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$ then equation 74 results.

74. $\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)d\tau \leftrightarrow F_1(\omega)F_2(\omega)$

To prove this theorem, we must use the Fourier Integral (equation 1) of f(t). Using substitution, equation 75 results.

75. $G(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\left[\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)d\tau\right]\mathrm{dt}$

We can change the order of integration which results in equation 76.

76. $= \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} e^{-i\omega t} f_2(t-\tau)dt\right]d\tau$

By the time shifting theorem (equation 22) we obtain equation 77.

77. $= \int_{-\infty}^{\infty} f_1(\tau)e^{-i\omega t}\, F_2(\omega)d\tau$

We can remove $F_2(\omega)$ from the integral because it is a constant in this case and it can clearly be seen that we obtain equation 78 which is our desired result[1].

78. $G(\omega) = F_1(\omega)F_2(\omega)$

Equation 79 is stating the frequency convolution theorem.

79. $f_1(t)f_2(t) \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(y)F_2(\omega - y)dy$

We can substitute this into the inversion formula (equation 2) to get the resulting, equation 80.

80. $= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(y)\, F_2(\omega - y)dy\right] d\omega$

We can change the order of the integration limits to obtain equation 81.

81. $= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, F_2(\omega - y)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(y)\, dy\right] d\omega$

From the time convolution theorem and the inversion formula we can clearly see that the resulting equation (equation 82) is our desired result and thus proving this statement made by the frequency convolution theorem[1].

82. $= f_1(t)f_2(t)$

<u>Sampling Theorem</u>

The sampling theorem is a technique that is important in transmission of information. This theorem is stating that if there is some frequency (ω_c) of which function f(t) is 0 above (Equation 83), then f(t) can be uniquely determined from its values of f_n (Equation 84)[1].

83. $F(\omega) = \ 0 \ \ for\ |\omega| \ \geq \ \omega_c$

84. $f_n = f\left(n\frac{\pi}{\omega_c}\right)$

To start this proof, we must insert our f_n function into the inversion formula (Equation 2) and change the integration limits because of the parameters set in equation 83. Equation 85 is showing the basic formula in this case with our f(t) (Equation 87) function where equation 86 is showing the same equation with our specific f_n in place[1].

85. $f(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} F(\omega)e^{i\omega t}\,d\omega$

86. $f_n = f\left(n\frac{\pi}{\omega_c}\right) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} F(\omega)e^{in\pi\omega/\omega_c}\,d\omega$

87. $f(t) = \sum_{n=-\infty}^{\infty} f_n \frac{\sin(\omega_c t - n\pi)}{\omega_c t - n\pi}$

Next we would like to expand this formula into a Fourier series (Equation 88) in the interval of $-\omega_c < \omega < \omega_c$.

88. $F(\omega) = \sum_{n=-\infty}^{\infty} A_n e^{-in2\pi\omega/2\omega_c}\,d\omega$ $\qquad A_n = \frac{1}{2\omega_c}\int_{-\omega_c}^{\omega_c} F(\omega)e^{in\pi\omega/\omega_c}\,d\omega$

We can conclude, from equation 86, through simple substitution that equation 89 will result.

89. $A_n = \frac{\pi}{\omega_c} f_n$

The sum (Equation 90) is then expressed as the periodic repetition of F(ω) and can be written as a product of F*(ω) and the pulse function $p_{\omega c.}$(Equation 91[1]).

90. $F*(\omega) = \sum_{n=-\infty}^{\infty} \frac{\pi}{\omega_c} f_n e^{in\pi\omega/\omega_c}$

91. $F(\omega) = p_{\omega_c}(\omega)\ \sum_{n=-\infty}^{\infty} \frac{\pi}{\omega_c} f_n e^{in\pi\omega/\omega_c}$

Our next step is to compare this to a Fourier Transform that we assume is true (Equation 92) and we can then see our expected result (Equation 87)[1].

92. $\frac{\omega_c}{\pi}\frac{\sin(\omega_c t - n\pi)}{\omega_c t - n\pi} \leftrightarrow p_{\omega_c}(\omega)\ f_n e^{in\pi\omega/\omega_c}$

These theorems are some of the many theorems that are used in the study of Fourier Transforms. There are a wide variety of fields that these transforms have applications on and this research is just some of the more basic results. With more extensive work and more complex theorems, there would be many other real world applications to be shown.

Bibliography

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Identities used

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt$$

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)dt = \phi(0)$$

$$\phi(t) = \int_{-\infty}^{\infty} \phi(x)\delta(t-x)dx$$

$$e^{iy} = \cos y + i \sin y$$

$$\frac{\omega_c}{\pi}\frac{\sin(\omega_c t - n\pi)}{\omega_c t - n\pi} \leftrightarrow p_{\omega_c}(\omega)\, f_n e^{in\pi\omega\,/\omega_c}$$