QUIZ 10 SOLUTIONS - MATH 251 Tuesday, November 20

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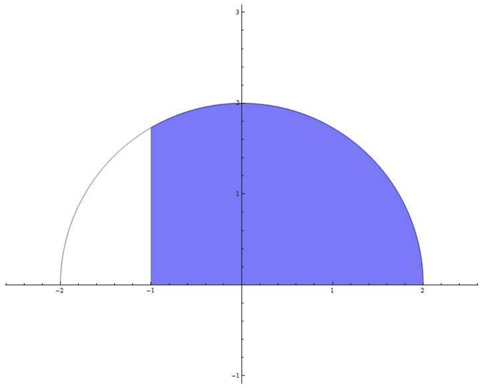
Read each problem **very carefully** before starting to solve it. Each problem is worth 5 points. It is necessary to show **all** your work. Correct answers without explanations are worth 0 points. GOOD LUCK!!

1. Consider $\displaystyle\int_{-1}^{2}\int_{0}^{\sqrt{4-x^2}}(x^2+y^2)dydx$.

   (a) Sketch the region of integration and carefully express it in polar coordinates.



To express the region we cut it in two pieces:

$$\mathcal{D}=\left\{(r,\theta):0\leq\theta\leq\frac{2\pi}{3},0\leq r\leq 2\right\}\cup\left\{(r,\theta):\frac{2\pi}{3}\leq\theta\leq 2\pi,0\leq r\leq-\frac{1}{\cos\theta}\right\}.$$

   (b) Use polar integration to calculate the value of the integral.

The integral over the second region requires $\int\sec^4\theta d\theta$ which we compute first to have it ready:

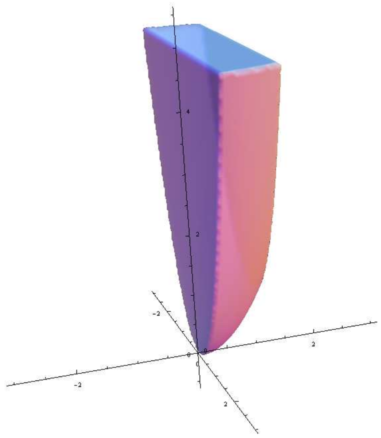
$$\begin{aligned}\int\sec^4\theta d\theta &= \int(\sec^2\theta)(\tan\theta)'d\theta \overset{\text{by parts}}{=} \sec^2\theta\tan\theta-\int\tan\theta(\sec^2\theta)'d\theta\\ &= \sec^2\theta\tan\theta-2\int\tan^2\theta\sec^2\theta d\theta \overset{u=\tan\theta}{=} \sec^2\theta\tan\theta-2\int u^2du\\ &= \sec^2\theta\tan\theta-\tfrac{2}{3}\tan^3\theta+C.\end{aligned}$$

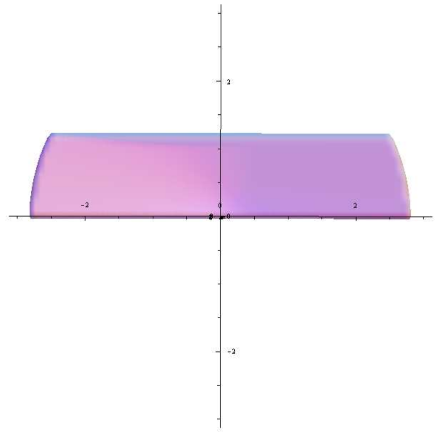
Now we have

$$\begin{aligned}&\int_{-1}^{2}\int_{0}^{\sqrt{4-x^2}}(x^2+y^2)dydx\\ &=\int_0^{2\pi/3}\int_0^2 r^2rdrd\theta+\int_{2\pi/3}^{2\pi}\int_0^{-1/\cos\theta}r^2rdrd\theta\\ &=\tfrac{2\pi}{3}\tfrac{r^4}{4}\big|_0^2+\tfrac{1}{4}\int_{2\pi/3}^{2\pi}\sec^4\theta d\theta\\ &=\tfrac{8\pi}{3}+(\sec^2\theta\tan\theta-\tfrac{2}{3}\tan^3\theta)\big|_{2\pi/3}^{2\pi}\\ &=\tfrac{8\pi}{3}+[0-((-2)^2(-\sqrt{3})-\tfrac{2}{3}(-\sqrt{3})^3)]\\ &=\tfrac{8\pi}{3}+2\sqrt{3}.\end{aligned}$$

2. Calculate $\iiint_{\mathcal{W}} y dV$, where $\mathcal{W}$ is the region above $z = x^2 + y^2$ and below $z = 5$ and bounded by $y = 0$ and $y = 1$.

We give a picture of the solid and the accompanying region of integration on the xy-plane.





The three-dimensional region is $\mathcal{W} = \{(x, y, z) : (x, y) \in \mathcal{D}, x^2 + y^2 \leq z \leq 5\}$.

The two-dimensional region $\mathcal{D}$ is $\mathcal{D} = \{(x, y) : 0 \leq y \leq 1, -\sqrt{5-y^2} \leq \sqrt{5-y^2}\}$.

Thus, we have

$$\begin{aligned}
\iiint_{\mathcal{W}} y dV &= \int_0^1 \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} \int_{x^2+y^2}^{5} y dz dx dy \\
&= \int_0^1 \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} y(5 - x^2 - y^2) dx dy \\
&= \int_0^1 (-\tfrac{1}{3} y x^3 - y^3 x + 5yx)\Big|_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} dy \\
&= 2\int_0^1 (-\tfrac{1}{3}\sqrt{5-y^2}^3 - y^2\sqrt{5-y^2} + 5\sqrt{5-y^2}) y dy \\
&= 2\int_0^1 (-\tfrac{1}{3}(5-y^2)^{3/2} - y^2(5-y^2)^{1/2} + 5(5-y^2)^{1/2}) y dy \\
&\overset{u=5-y^2}{=} 2\int_5^4 (-\tfrac{1}{3}u^{3/2} - (5-u)u^{1/2} + 5u^{1/2})(-\tfrac{1}{2} du) \\
&= \int_4^5 \tfrac{2}{3} u^{3/2} du \\
&= \tfrac{4}{15}\sqrt{u}^5\Big|_4^5 \\
&= \tfrac{4}{15}(\sqrt{5}^5 - 32).
\end{aligned}$$