EXAM 1 SOLUTIONS - CSCI 341

Friday, September 21

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Read each problem **very carefully** before starting to solve it. Each problem is worth 10 points. It is necessary to show **all** your work. Correct answers without explanations are worth 0 points. GOOD LUCK!!

1. (a) Show that $A \to \neg B \equiv \neg(A \wedge B)$

A	B	$\neg B$	$A \to \neg B$	$A \wedge B$	$\neg(A \wedge B)$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	T

(b) The contrapositive of $A \to B$ is

$$\neg B \to \neg A$$

(c) The negation of the statement "George is a Democrat or George is a Republican" is the statement

"George is neither a Democrat nor a Republican".

(d) The converse of the statement "If there are more than 50 Republican Senators, then there are at most 49 Democratic Senators" is the statement

"If there are at most 49 Democratic Senators,
then there are more than 50 Republican Senators".

(e) The statement "If $6 \mid x$ then $4 \mid x$" is False

Proof: By counterexample: For $x = 6$, $6 \mid 6$ but $4 \nmid 6$.

(f) Prove the statement "If $x^2 - 2x + 3$ is odd, then x is even", where x is assumed to be an integer.

Proof: By contraposition, we show

If x is odd, then $x^2 - 2x + 3$ is even.

Suppose x is odd.
Then $x = 2k + 1$, for some $k \in \mathbb{Z}$.
So

$$\begin{aligned} x^2 - 2x + 3 &= (2k+1)^2 - 2(2k+1) + 3 \\ &= 4k^2 + 4k + 1 - 4k - 2 + 3 \\ &= 4k^2 + 2 = 2(2k^2 + 1). \end{aligned}$$

Since $2k^2 + 1 \in \mathbb{Z}$, this shows that $x^2 - 2x + 3$ is even.

2. Fill in the blanks:

   (a) $\{a, b\} \ \in \ \{a, \{b\}, \{a, b\}\}$;

   (b) $\{0, 1, 2, 3, 4, 5\} \cap (\{0, 3, 4, 5, 6, 7\} - \{0, 1, 2, 3\}) = \{0, 1, 2, 3, 4, 5\} \cap \{4, 5, 6, 7\} = \{4, 5\}$;

   (c) $\mathcal{P}(\{0, a\}) = \{\emptyset, \{0\}, \{a\}, \{0, a\}\}$;

   (d) $A - B = \{x \in A : x \notin B\}$;

   (e) $A \oplus B = \{x : (x \in A \text{ or } x \in B) \text{ and } x \notin A \cap B\}$;

   (f) Let $A = \{2k + 7 : k \in \mathbb{Z}\}$ and $B = \{4k + 3 : k \in \mathbb{Z}\}$. Show that $B \subset A$.
   Proof:
   This consists of two parts:

Part 1: First show that $B \subseteq A$:
Let $x \in B$.
Then $x = 4k + 3$, for some $k \in \mathbb{Z}$.
So $x = 4k - 4 + 7 = 2(2k - 2) + 7$.
Since $2k - 2 \in \mathbb{Z}$, we get that $x \in A$.

Part 2: Second show that $B \neq A$:
We have $9 \in A$, since $9 = 2 \cdot 1 + 7$.
On the other hand $9 \notin B$, since $4k + 3 = 9$ has no solution in $\mathbb{Z}$.

3. Fill in the blanks:

(a) $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$;

(b) $\mathsf{cons}(\mathsf{head}(\langle\langle a\rangle, \langle\rangle, \langle a, b\rangle\rangle), \mathsf{tail}(\mathsf{tail}(\langle\langle a\rangle, \langle a, b\rangle, \langle c\rangle, \langle\rangle, \langle a, b\rangle\rangle)))$

$$\begin{aligned} &= \mathsf{cons}(\langle a\rangle, \mathsf{tail}(\langle\langle a, b\rangle, \langle c\rangle, \langle\rangle, \langle a, b\rangle\rangle)) \\ &= \mathsf{cons}(\langle a\rangle, \langle\langle c\rangle, \langle\rangle, \langle a, b\rangle\rangle) \\ &= \langle\langle a\rangle, \langle c\rangle, \langle\rangle, \langle a, b\rangle\rangle. \end{aligned}$$

(c) $\{\Lambda, abab, aabbabab, aaabbbababab, aaaabbbbabababab, \ldots\} = \{a^n b^n (ab)^n : n \geq 0\}$;

(d) Only in (d), assume $L = \{\Lambda, a, bab\}$ and $M = \{aba, b, bab\}$.

$$LM = \{aba, b, bab, aaba, ab, abab, bababa, babb, babbab\}.$$

(e) $L^+ = L^1 \cup L^2 \cup L^3 \cup L^4 \cup \cdots$;

(f) The statement that for all languages L and M,

$$L^* - M^* = (L - M)^*$$

is False

Proof: For any languages L and M,

$$\Lambda \in L^*, \qquad \Lambda \in M^*, \qquad \Lambda \in (L - M)^*.$$

Therefore, $\Lambda \notin L^* - M^*$, but $\Lambda \in (L - M)^*$.
So $L^* - M^* \neq (L - M)^*$.

4. Fill in the blanks:

(a) If $f : A \to B$ and $S \subseteq A$,
$$f(S) = \{f(s) : s \in S\};$$

(b) If $f : A \to B$ and $T \subseteq B$,
$$f^{-1}(T) = \{a \in A : f(a) \in T\};$$

(c) Finish the formal statement of the division algorithm:

For every integers m and n, with $n \neq 0$, there exist unique $q, r \in \mathbb{Z}$, with $0 \leq r < |n|$, such that
$$m = n \cdot q + r.$$

(d) Apply Euclid's algorithm to find the gcd of 612 and 50. Show carefully all iterations of the algorithm:

Iteration 1: $612 = 50 \cdot 12 + 12$;

Iteration 2: $50 = 12 \cdot 4 + 2$;

Iteration 3: $12 = 2 \cdot 6 + 0$.

So $\mathsf{gcd}(612, 50) = 2$.

(e) $\mathsf{dist}(0, \mathsf{map}(+)(\mathsf{pairs}(\mathsf{seq}(3), \mathsf{seq}(3))))$

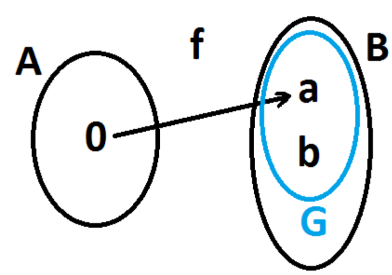
$$\begin{aligned} &= \mathsf{dist}(0, \mathsf{map}(+)(\mathsf{pairs}(\langle 0,1,2,3\rangle, \langle 0,1,2,3\rangle))) \\ &= \mathsf{dist}(0, \mathsf{map}(+)(\langle (0,0),(1,1),(2,2),(3,3)\rangle)) \\ &= \mathsf{dist}(0, \langle 0,2,4,6\rangle) \\ &= \langle (0,0),(0,2),(0,4),(0,6)\rangle. \end{aligned}$$

(f) The statement that, for every function $f : A \to B$ and every subset $G \subseteq B$,
$$f(f^{-1}(G)) = G$$

is <u>False</u>

Proof:

We present a counterexample.
Let $A = \{0\}$, $B = \{a, b\}$.
Consider the function $f : A \to B$ defined by $f(0) = a$ and let $G = B = \{a, b\}$.
Then we have $f(f^{-1}(G)) = f(f^{-1}(\{a, b\})) = f(\{0\}) = \{a\} \neq G$.



5. (a) A function $f : A \to B$ is injective (or 1-1) if

$$\text{for all } a, a' \in A, \text{ if } a \neq a', \text{ then } f(a) \neq f(a').$$

(b) A function $f : A \to B$ is surjective (or onto) if

$$\text{for all } b \in B, \text{ there exists an } a \in A, \text{ such that } f(a) = b.$$

Equivalently, if $f(A) = B$.

(c) Consider the function $f : \mathbb{Z}_6 \to \mathbb{Z}_6$, defined by $f(x) = 5x \mod 6$.

x	$f(x)$
0	0
1	5
2	4
3	3
4	2
5	1

(i) The statement "f is injective" is True, because no two distinct elements in the domain map to the same element in the codomain.

(ii) The statement "f is surjective" is True, because all six elements in the codomain are in the image of f.

(iii) The statement "f has an inverse" is True, because f is bijective (recall that we proved that a function f has an inverse if and only if it is bijective).

(d) Let $f : A \to B$ and $g : B \to C$ be functions. The statement "If $g \circ f$ is surjective, then g is surjective" is True

Proof:

Let $c \in C$. Then, since $g \circ f : A \to C$ is surjective, there exists $a \in A$, such that

$$g(f(a)) = c.$$

But then, there exists $b = f(a) \in B$, such that

$$g(b) = g(f(a)) = c.$$

This proves that $g : B \to C$ is surjective.