EXAM 2 SOLUTIONS - CSCI 341

Friday, October 12

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Read each problem **very carefully** before starting to solve it. Each problem is worth 10 points. It is necessary to show **all** your work. Correct answers without explanations are worth 0 points. GOOD LUCK!!

1. (a) The sets A and B have the **same cardinality** if, by definition,

$$\text{there exists a bijection } f : A \to B.$$

(b) A set A is countably infinite if, by definition,

$$|A| = |\mathbb{N}|, \text{ i.e., there exists a bijection } f : \mathbb{N} \to A.$$

(c) The set $\underline{\mathcal{P}(A)}$ has cardinality greater than $|A|$.

(d) Give a bijection $f : \mathbb{N} \to S$, where S is the set of all strings over $A = \{a\}$ that have odd length. (You do not have to prove that it is a bijection.)

$$f(n) = a^{2n+1}, \quad \text{for all } n \in \mathbb{N}.$$

(e) Prove that the set $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$ is countably infinite.

Proof:

Define a function $f : \mathbb{N} \to A$ by $f(n) = \frac{1}{n+2}$, for all $n \in \mathbb{N}$.

f is injective: Suppose $f(n) = f(m)$. Then $\frac{1}{n+2} = \frac{1}{m+2}$. Therefore, $n + 2 = m + 2$ and it follows that $n = m$. So f is injective.

f is surjective: Suppose that $\frac{1}{k} \in A$, with $k \geq 2$. Then, we have $f(k-2) = \frac{1}{(k-2)+2} = \frac{1}{k}$. Thus, since $k - 2 \in \mathbb{N}$, we get that f is surjective.

Hence $f : \mathbb{N} \to A$ is a bijection, showing that A is countably infinite.

(f) The statement "If a set A is countably infinite and there exists an injection $f : A \to B$, then B is countably infinite" is <u>false</u>

Proof:

Consider the set A of Part (e). The function $f : A \to (0, 1)$ given by

$$f(x) = x, \text{ for all } x \in A,$$

is definitely injective and we showed that A is countably infinite. However, we know (by diagonalization) that $(0, 1)$ is uncountable.

2. (a) Give an inductive definition of the set $A = \{3k + 5 : k \in \mathbb{N}\}$.

Basis: $5 \in A$;

Induction: If $a \in A$, then $a + 3 \in A$.

(b) Give an inductive definition of the set C of all lists of odd length over $A = \{a, b\}$.

Basis: $\langle a \rangle, \langle b \rangle \in C$;

Induction: If $L \in C$, then $a :: a :: L$, $a :: b :: L$, $b :: a :: L$, $b :: b :: L \in C$.

(c) A set is defined inductively as follows:
Basis: $1 \in S$;
Induction: If $x \in S$, then $1 + \frac{1}{x} \in S$.
Write at least seven of the elements to give a flavor of S:

$$S = \{1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots\}.$$

(d) Give an inductive definition of the set

$$A = \{4, 7, 10, 12, \ldots\} \times \{3, 9, 27, 81, \ldots\}.$$

Basis: $(4, 3) \in A$;

Induction: If $(x, y) \in A$, then $(x + 3, y)$, $(x, 3y) \in A$.

(e) Write an inductive definition of $K = \{a^{2n} : n \in \mathbb{N}\} \cup \{b^{3n+2} : n \in \mathbb{N}\}$.

Basis: $\Lambda, bb \in K$;

Induction: If $x \in K$ then if x starts with a b then $bbbx \in K$ else $aax \in K$.

(f) Consider the following set S of strings over alphabet $A = \{a, b\}$:

$$S = \{x \in \{a, b\}^* : x \text{ has the same number of } a\text{'s and } b\text{'s}\}.$$

The statement "S is defined by the following inductive definition:
Basis: $\Lambda \in S$;
Induction: If $x \in S$, then $abx, bax, axb, bxa \in S$"
is <u>false</u>
Proof:
The string $s = aabbbbaa$ is certainly in S. However, there is no way to build up s inductively using the inductive definition above.

3. (a) Give a recursive definition of the function $\mathsf{length} : \{a,b\}^* \to \mathbb{N}$, defined by

$$f(x) = \text{the length of } x, \quad \text{for all } x \in \{a,b\}^*.$$

Basis: $f(\Lambda) = 0$;

Recursion: $f(ax) = 1 + f(x)$ and $f(bx) = 1 + f(x)$.

(b) Give a recursive definition of the function $\mathsf{dist} : A \times \mathsf{Lists}[B] \to \mathsf{Lists}[A \times B]$, defined by

$$\mathsf{dist}(a, \langle b_1, b_2, \ldots, b_n \rangle) = \langle (a,b_1), (a,b_2), \ldots, (a,b_n) \rangle.$$

Basis: $\mathsf{dist}(a, \langle\rangle) = \langle\rangle$, for all $a \in A$;

Recursion: $\mathsf{dist}(a, L) = (a, \mathsf{head}(L)) :: \mathsf{dist}(a, \mathsf{tail}(L))$, if $L \neq \langle\rangle$, for all $a \in A$.

(c) Give a recursive definition of a function $\mathsf{ins} : \mathbb{R} \times \mathsf{Lists}[\mathbb{R}] \to \mathsf{Lists}[\mathbb{R}]$ that is is supposed to operate as follows: Upon taking a real number x and a list L of real numbers that is ordered in decreasing order, it is supposed to insert x in the list and output a new ordered list with the numbers still in decreasing order. To resolve conflicts the function inserts duplicates on the left of already existing ones.

Basis: $\mathsf{ins}(x, \langle\rangle) = \langle x \rangle$, for all $x \in \mathbb{R}$;

Recursion: $\mathsf{ins}(x, L) =$ if $x < \mathsf{head}(L)$ then $\mathsf{ins}(x, \mathsf{tail}(L))$ else $x :: L$, for $L \neq \langle\rangle$ and all $x \in \mathbb{R}$.

(d) Give a recursive definition of the function $\mathsf{Apply} : (\mathbb{N} \to \mathbb{N}) \times \mathsf{Lists}[\mathbb{N}] \to \mathsf{Lists}[\mathbb{N}]$, defined as follows:

$$\mathsf{Apply}(f, \langle x_0, x_1, \ldots, x_n \rangle) = \langle f(x_0), f(x_1), \ldots, f(x_n) \rangle.$$

Basis: $\mathsf{Apply}(f, \langle\rangle) = \langle\rangle$;

Recursion: $\mathsf{Apply}(f, L) = f(\mathsf{head}(L)) :: \mathsf{Apply}(f, \mathsf{tail}(L))$, if $L \neq \langle\rangle$.

(e) In this part, you may use, if you decide it is convenient to do so, the function Apply that you defined in Part (e), in the spirit of incrementally building computer code by reusing previously defined functions and procedures.

Give a recursive definition of the function
$\mathsf{SpecApply} : (\mathbb{N} \to \mathbb{N}) \times \mathsf{Lists}[\mathbb{N}] \to \mathsf{Lists}[\mathbb{N}]$, defined as follows:

$$\mathsf{SpecApply}(f, \langle x_0, x_1, \ldots, x_n \rangle) = \langle x_0, f(x_1), f(f(x_2)), f(f(f(x_3))), \ldots, f^n(x_n) \rangle.$$

Note that $f^i(x)$ denotes i-fold composition of f with itself and not i-th power.

Basis: $\mathsf{SpecApply}(f, \langle\rangle) = \langle\rangle$;

Recursion: $\mathsf{SpecApply}(f, L) = \mathsf{head}(L) :: \mathsf{Apply}(f, \mathsf{SpecApply}(f, \mathsf{tail}(L)))$, if $L \neq \langle\rangle$.
Note that

$$\begin{aligned}
\mathsf{SpecApply}(f, \langle x_0, x_1, \ldots, x_n \rangle) &= \langle x_0, f(x_1), f^2(x_2), \ldots, f^n(x_n) \rangle \\
&= x_0 :: \langle f(x_1), \ldots, f^n(x_n) \rangle \\
&= x_0 :: \mathsf{Apply}(f, \langle x_1, f(x_2), \ldots, f^{n-1}(x_n) \rangle) \\
&= x_0 :: \mathsf{Apply}(f, \mathsf{SpecApply}(f, \langle x_1, \ldots, x_n \rangle)).
\end{aligned}$$

4. (a) A **grammar** is a tuple $G = \langle N, T, S, P \rangle$, where:

   (i) N is a set of nonterminals;

   (ii) T is a set of terminals;

   (iii) S is the start symbol, with $S \in N$;

   (iv) P is a set of productions, i.e., rules of the form $\alpha \to \beta$, with $\alpha, \beta \in (N \cup T)^*$.

(b) Let $G = \langle N, T, S, P \rangle$ be the grammar given in our adopted shorthand notation as follows

$$\begin{aligned} R &\to AB \\ A &\to Aa|a \\ B &\to Bb|\Lambda \end{aligned}$$

Give formally each of the four components of the grammar:

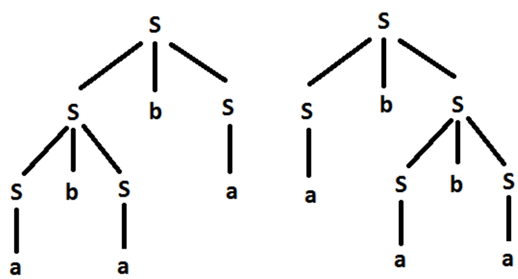
   (i) $N = \{R, A, B\}$;

   (ii) $T = \{a, b, \Lambda\}$;

   (iii) $S = R$;

   (iv) $P = \{R \to AB, A \to Aa, A \to a, B \to Bb, B \to \Lambda\}$.

(c) The statement "The grammar $S \to a|SbS$ is ambiguous" is <u>true</u>
Proof:



(d) Give a grammar for the language $\{a^m b^n : m, n \in \mathbb{N}, n > 0\}$.

$$S \to aS|Sb|b$$

(e) Give a grammar for the even palindromes over $\{a, b, c\}$.

$$S \to aSa|bSb|cSc|\Lambda$$

(f) Give a grammar for the language $\{a^n bc^n : n \in \mathbb{N}\}^*$.

$$\begin{aligned} S &\to AS|\Lambda \\ A &\to aAc|b \end{aligned}$$

5. (a) The powerset $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$;

(b) The powerset $\mathcal{P}(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{0,1\}, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$;

(c) Every element in $\mathcal{P}(\{1,2\})$ "gives rise" to two elements in $\mathcal{P}(\{0,1,2\})$. Describe in which way this happens.

Each subset of $\{1,2\}$ gives rise to the same set (that does not contain 0) or to the union of the same set with $\{0\}$.

(d) Use the Principle of Induction to prove that the following property $P(n)$ holds for all $n \in \mathbb{N}$:

$$P(n): \qquad 1 + 3 + 3^2 + 3^3 + \cdots + 3^n = \frac{1}{2}(3^{n+1} - 1).$$

Basis: $1 = \frac{1}{2}(3^1 - 1)$;

Induction Hypothesis: $1 + 3 + 3^2 + \cdots + 3^k = \frac{1}{2}(3^{k+1} - 1)$;

Induction Step:

$$\begin{aligned} 1 + 3 + 3^2 + 3^k + 3^{k+1} &= \tfrac{1}{2}(3^{k+1} - 1) + 3^{k+1} \\ &= \tfrac{1}{2}(3^{k+1} - 1 + 2 \cdot 3^{k+1}) \\ &= \tfrac{1}{2}(3 \cdot 3^{k+1} - 1) \\ &= \tfrac{1}{2}(3^{(k+1)+1} - 1). \end{aligned}$$

(e) (In this part use for your benefit Part (c). Recall $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$.)

Use the Principle of Induction to prove that the following property $P(n)$ holds for all $n \in \mathbb{N}$:

$$P(n): \qquad |\mathcal{P}(\mathbb{Z}_n)| = 2^n.$$

Basis: $|\mathcal{P}(\emptyset)| = 2^0 = 1$ holds since $\mathcal{P}(\emptyset) = \{\emptyset\}$;

Induction Hypothesis: $|\mathcal{P}(\mathbb{Z}_k)| = 2^k$;

Induction Step: Every subset of $\mathbb{Z}_k$ gives rise to two subsets of $\mathbb{Z}_{k+1}$, itself and the union of itself and $\{k\}$. Therefore, since there are 2^k subsets of $\mathbb{Z}_k$, we get that

$$|\mathcal{P}(\mathbb{Z}_{k+1})| = 2 \cdot |\mathcal{P}(\mathbb{Z}_k)| = 2 \cdot 2^k = 2^{k+1}.$$