QUIZ 4 SOLUTIONS - CSCI 341

Friday, September 28

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Read each problem **very carefully** before starting to solve it. Each problem is worth 5 points. It is necessary to show **all** your work. Correct answers without explanations are worth 0 points. GOOD LUCK!!

1. Fill in the missing information (please, try to be formal, precise and concise):

   (a) The sets A and B have **the same cardinality** if

   $$\text{there exists a bijection } f : A \to B.$$

   (b) A set A is defined to be **countably infinite** if

   $$|A| = |\mathbb{N}|.$$

   (c) Cantor's Theorem: For any set A,

   $$|A| < |\mathcal{P}(A)|.$$

   (d) Cantor's Theorem is proved by

   $$\text{diagonalization.}$$

   (e) The same technique (as that named in (d)) is also used to show, e.g., that

   $$\text{the set } (0,1) \text{ is uncountable}$$

   and, also, that

   $$\text{the set } \mathbb{N} \to \mathbb{N} \text{ is uncountable.}$$

2. Let $A = \{x \in \mathbb{N} : x \mod 7 = 5\}$. Prove (without skipping any details) that A is countably infinite.

   We must show that there exists a bijection $f : \mathbb{N} \to A$. We define, for all $n \in \mathbb{N}$,

   $$f(n) = 7n + 5.$$

   - f is injective: Suppose that, for $n, m \in \mathbb{N}$, $f(n) = f(m)$.
     Then, we get $7n + 5 = 7m + 5$.
     Subtract 5 and divide by 7 to get $n = m$.
     Thus, $f : \mathbb{N} \to A$ is injective.
   - f is surjective: Suppose $x \in A$, i.e., that $x \mod 7 = 5$.
     Then, by the division algorithm $x = 7q + 5$, for some $q \in \mathbb{N}$.
     It follows that $f(q) = 7q + 5 = x$.
     Thus $f : \mathbb{N} \to A$ is also surjective.

   Since $f : \mathbb{N} \to A$ is a bijection, we conclude that A is countably infinite.

3. Consider the alphabet A of all symbols allowed in Java® programs. Give the characteristics (not asking for exact numbers).

   (a) The cardinality of A is finite

   (b) The cardinality of A^* is countably infinite because

   it is the union of countably many finite sets.

   (c) The cardinality of all valid (or correct) Java programs is countably infinite, because

   it is a subset of A^*.

   (d) The cardinality of the set $\mathcal{P}(A^*)$ of all languages over A is uncountable by Cantor's Theorem

   (e) From (c) and (d) we can conclude that

   there exists languages over A which cannot be decided by any Java programs, because there are more languages over A than there are Java programs.