EXAM 1 SOLUTIONS - MATH 300
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Problem 1 *Use Boole's method of equational reasoning to establish the validity of the following argument:*

$$\begin{array}{l} A'(B' \cup C') = 0 \\ CD' = 0 \\ DE = 0 \\ \therefore\ A'E = 0 \end{array}$$

Solution: We have the following

$$\begin{array}{rcll} A'E & = & A'1E & \text{(Intersection with Universe)} \\ & = & A'(B \cup B')E & (B \cup B' = 1) \\ & = & A'BE \cup A'B'E & \text{(Distributivity)} \\ & = & A'B1E & \text{(Hypothesis 1)} \\ & = & A'B(C \cup C')E & (C \cup C' = 1) \\ & = & A'BCE \cup A'BC'E & \text{(Distributivity)} \\ & = & A'BC1E & \text{(Hypothesis 1)} \\ & = & A'BC(D \cup D')E & (D \cup D' = 1) \\ & = & A'BCDE \cup A'BCD'E & \text{(Distributivity)} \\ & = & 0 \cup 0 & \text{(Hypotheses 3 and 2)} \\ & = & 0 & (0 \cup 0 = 0) \end{array}$$

■

Problem 2 *Use Carroll's tree method to show the validity of the following argument:*

$$\begin{array}{l} ABF = 0 \\ ACD = 0 \\ B'D'E' = 0 \\ C'DE' = 0 \\ D'E'F' = 0 \\ \therefore\ AE' = 0 \end{array}$$

Solution: The Carroll tree of Figure 1 justifies the validity of the given argument, with the red numbers referring to the hypotheses that justify why the expressions labeling the leaves correspond to the empty set. ■

Problem 3 *Determine whether the following argument is valid*

$$\begin{array}{l} P \to Q \\ (Q \vee R) \wedge \neg(Q \wedge R) \\ \therefore\ \neg Q \to (\neg P \wedge R) \end{array}$$

Solution: Suppose that $\mathbf{e}$ is an assignment of truth values to the propositional variables that evaluates $P \to Q$ and $(Q \vee R) \wedge \neg(Q \wedge R)$ to value true.

If $\mathbf{e}(Q) = 1$, then $\neg Q \to (\neg P \wedge R)$ is evaluated to true and there is nothing to prove.

If, on the other hand, $\mathbf{e}(Q) = 0$, then, since $P \to Q$ is evaluated to true, we must have $\mathbf{e}(P) = 0$. Moreover, since $(Q \vee R) \wedge \neg(Q \wedge R)$ is evaluated to true, we must also have $\mathbf{e}(R) = 1$. But, then, $\neg P \wedge R$ is also evaluated to true, showing that $\neg Q \to (\neg P \wedge R)$ is evaluated to true.

The reasoning above shows that the given argument is a valid argument. ■

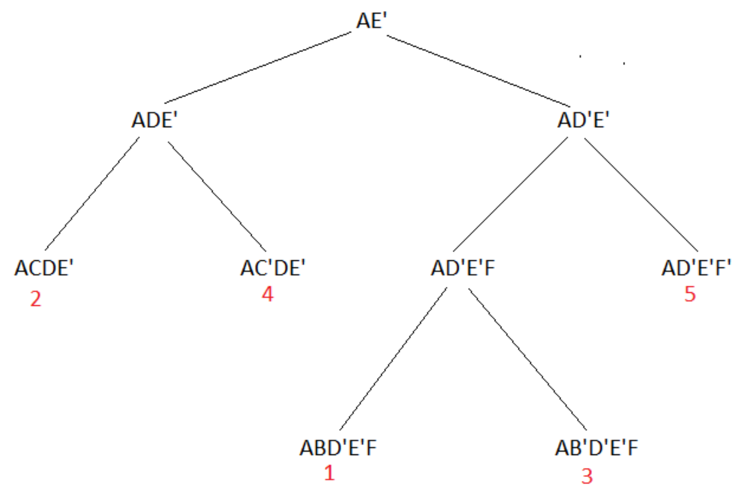

Figure 1: Carroll's Tree for Problem 2.

Problem 4 *In this problem, we are going to use the following notation:*

N	*"needs to sleep 12 hours"*
L	*"is the life of a party"*
S	*"is a skier"*
J	*"likes junk food"*
T	*"likes TV soap operas"*
M	*"rides a motorcycle".*

Convert the statements in the following argument into propositional formulas and, then, determine whether the argument is valid, if the universe of discourse is the set of all people:

Those who do not need 12 hours sleep are the life of a party.
Those who need 12 hours sleep are not skiers nor do they like junk food.
Anyone who is not a skier likes TV soap operas.
A person who is the life of a party does not like TV soap operas nor ride a motorcycle.
Skiers do not like junk food.
$\therefore$ *Everyone likes TV soap operas and rides a motorcycle.*

Solution: Using the notation suggested at the beginning of the problem, the given argument may be recast in propositional logic as follows:

$$\begin{array}{l} \neg N \rightarrow L \\ N \rightarrow (\neg S \wedge \neg J) \\ \neg S \rightarrow T \\ L \rightarrow (\neg T \wedge \neg M) \\ S \rightarrow \neg J \\ \therefore\ T \wedge M \end{array}$$

The displayed argument is not valid. Consider the truth assignment **e**, whose values on the propositional variables are summarized in the following table:

Variable	N	S	L	T	M	J
e-Assignment	0	1	1	0	0	0

Under this assignment, it is not difficult to see that all hypotheses in the argument above are evaluated to truth value true, but the conclusion is evaluated to truth value false. Thus, the given argument is not a valid argument. ■

Problem 5 (**An Application of Compactness**)

> **König's Infinity Lemma:** *If a tree contains infinitely many vertices, each having finitely many children, then it has at least one infinite path.*

Use carefully the Compactness Theorem to prove König's Lemma.
(**Hint:** *Introduce for every vertex v in the tree a propositional variable P_v. The intuition is that P_v will be assigned the truth value 1 if and only if v is in the infinite path whose existence is asserted in the conclusion of the statement.*)

Solution: We introduce, as suggested in the Hint, for every vertex v in the tree a propositional variable P_v. The intuition is that P_v will be assigned the truth value 1 if and only if v is in the infinite path whose existence is asserted in the conclusion of the statement. We consider the infinite collection $\mathcal{S}$ of propositional formulas that consists of the following types of formulas:

(a) P_{v_0}, where v_0 is the root of the given tree;

(b) $P_{v_1} \vee P_{v_2} \vee \cdots \vee P_{v_{k_n}}$, where $v_1, v_2, \ldots, v_{k_n}$ are all vertices of the tree at level n, i.e., those that are connected to the root by a path of length n;

(c) $P_v \to \neg P_u$, if, either $u \neq v$ are at the same level in the tree, or u is one level higher than v and (v, u) is not an edge of the tree, i.e., u is not a child of v.

The single formula of Type (a) says that the root must necessarily be a vertex on the infinite path. Formulas of Type (b) say that at least one vertex at each level must be on the infinite path and formulas of Type (c) say that no two different vertices at the same level must be on the path and, moreover, no two vertices not joined by an edge can be successive vertices on the path.

We implement the following proof strategy:

- We first show that every finite subset $\mathcal{S}_0$ of $\mathcal{S}$ is satisfiable;
- By Compactness, we may now infer that $\mathcal{S}$ is satisfiable (this step does not require proof);
- We prove that the satisfiability of $\mathcal{S}$ implies the existence of an infinite path in the tree.

Every finite subset $\mathcal{S}_0$ of $\mathcal{S}$ is satisfiable: Let n be an integer large enough that all variables appearing in the finite $\mathcal{S}_0$ refer to vertices at level at most n. Consider a vertex v_n at that level n and the unique path $v_0, v_1, v_2, \ldots, v_n$ joining the root v_0 with v_n. Define the assignment $\mathbf{e}$ of truth values to the propositional variables appearing in $\mathcal{S}_0$ defined as follows:

$$\mathbf{e}(P_v) = 1 \quad \text{iff} \quad v = v_i, \text{ for some } i = 0, 1, \ldots, n.$$

Clearly, all axioms of Types (a)-(c) that are in $\mathcal{S}_0$ are satisfied by $\mathbf{e}$, i.e., $\mathcal{S}_0$ is satisfiable. □

If $\mathcal{S}$ is satisfiable, then there exists an infinite path: Suppose $\mathbf{e}$ is a satisfying assignment for the entire set $\mathcal{S}$. We form the path $v_0, v_1, v_2, \ldots$ where v_i is, by the satisfiability of formulas of Type (b) and (c), the unique vertex at level i for which $\mathbf{e}(P_{v_i}) = 1$. The fact that $\mathbf{e}(P_{v_0}) = 1$, i.e., that the root is included, is ensured by Formula (a). The fact that the path is infinite is ensured by the formulas of Type (b). Finally, the fact that v_{n+1} is a child of v_n in the tree is ensured by the formulas of Type (c). □

This concludes the proof of König's Lemma, i.e., of the existence of an infinite path in a tree with infinitely many vertices, each of which has finitely many children. ■