HOMEWORK 5 - MATH 300 Due: Monday, April 8

YOUR NAME:________________________ George Voutsadakis

Read each problem **very carefully** and try to understand it well before starting to solve it. It is necessary to show **all** your work. Correct answers without explanations are worth 0 points. Write your own solutions and be neat!! **Take pride in your work!! Do not hand in scratchy doodles.**

1. Consider the directed graph $\mathbf{G}$ given by the binary relation "is less than" on the set of numbers $G = \{0, 1, 2, 3, 4\}$.
   (a) Express $\mathbf{G}$ as a set of ordered pairs.
   (b) Give a table for $\mathbf{G}$.
   (c) Draw $\mathbf{G}$.
2. Consider the directed graph $\mathbf{G}$ given by the binary relation "divides" on the set of numbers $G = \{0, 1, 2, \ldots, 10\}$.
   (a) Express $\mathbf{G}$ as a set of ordered pairs.
   (b) Give a table for $\mathbf{G}$.
   (c) Draw $\mathbf{G}$.
3. For the following formulas, state clearly whether they are atomic or non-atomic, create the (formula) syntax tree and apply carefully (step-by-step) the recursive procedure for finding all subformulas:
   (a) $\forall y(r_2 f_1 xy)$
   (b) $r_3 f_3 yzzyf_1 z$
   (c) $\exists z((r_1 z) \leftrightarrow (r_1 f_2 yz))$
   (d) $\forall z((r_2 zx) \to \exists y((r_2 f_1 xy) \to ((r_1 z) \to \forall x(r_1 f_1 x))))$
4. This exercise (Textbook 5.1.2), due to the concepts that it examines, is very important in understanding first-order logic. Thus, you should make sure to study and understand the concepts involved in some depth.

   Do the following for each first-order formula below: (a) Indicate, for each variable involved, its bound and its free occurrences. (b) Tell clearly the scope of each quantifier. (c) For every bound occurrence of a variable, indicate which occurrence of a quantifier binds it.
   (i) $\forall y \exists y((r_2 xz) \vee \exists z(r_2 xz))$
   (ii) $\forall x(\forall z(((r_1 y) \wedge \neg(r_2 zy)) \vee \exists y(r_3 yxz)) \wedge (r_2 yz))$
   (iii) $(\exists y(r_3 xyz)) \to \exists x((\forall x(r_3 zyy)) \to (r_1 f_3 xyx))$
5. This exercise involves formulas over the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ of the natural numbers $\mathbb{N}$. For each string, state (with a very brief explanation, when appropriate) whether it is a formula. If yes, state clearly whether it is atomic, whether it is a sentence, build its (formula) syntax tree and find all its subformulas.
   (a) $1 \cdot 1 \approx 1 + 1$
   (b) $(1 + 1 < 1) \approx 0$
   (c) $(x \cdot (x \approx y) \cdot z) \leftrightarrow (1 < (x + (y \cdot z)))$
   (d) $\forall x(x + x \approx (1 + 1) \cdot x)$
   (e) $\forall z(((z \approx x) \wedge (y < x)) \vee \exists y \exists z(z \approx (z \cdot y)))$

6. This exercise involves formulas over the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ of the natural numbers $\mathbb{N}$. Do the following for each formula below: (a) Indicate, for each variable involved, its bound and its free occurrences. (b) Tell clearly the scope of each quantifier. (c) For every bound occurrence of a variable, indicate which occurrence of a quantifier binds it.
   (i) $\neg(\forall x(x \approx y) \leftrightarrow \exists x((y \approx x) \wedge (x < z)))$
   (ii) $\forall y((z < (x \cdot y)) \rightarrow ((z \approx z) \wedge \forall x(y < x)))$
   (iii) $(\forall x \neg(\exists x(x < y) \vee \neg((x < y) \wedge \exists x(y < x)))) \wedge (z \approx z)$
7. Translate the following first-order sentences over the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ into English:
   (a) $\forall x(\exists y(x \approx y + y + 1) \rightarrow \exists y(x + 1 \approx y + y))$
   (b) $\forall x((\exists y(x \approx 3 \cdot y) \wedge \exists y(x \approx 5 \cdot y)) \rightarrow \exists y(x \approx 15 \cdot y))$
   (c) $\forall x(x \approx 0 \vee \forall y((\neg(y < x) \wedge \neg(y \approx x)) \rightarrow \neg\exists z(x \approx z \cdot y)))$
   (d) $\exists x \exists y(\neg(x \approx y) \wedge \exists w(x \approx 7 \cdot w) \wedge \exists w(y \approx 7 \cdot w) \wedge$
   $\forall z((\exists w(x \approx z \cdot w) \wedge \exists w(y \approx z \cdot w)) \rightarrow ((z < 7) \vee (z \approx 7))))$

   Important Note: This problem is not asking you to simply *transliterate* first-order into English. For example, "there exists x, such that for all y, x is less than y or x is equal to y" is the transliteration of $\exists x \forall y((x < y) \vee (x \approx y))$, but **not** its English *translation*. Rather, an English translation, i.e., a sentence that would naturally express in English what this sentence means, is "there exists a smallest element x". Thus, even though transliteration is an almost mechanical phonetic process, translation requires some understanding of the meaning of the sentence in the intended interpretation structure(s).
8. Write first-order formulas over $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ that express the following relations on $\mathbb{N}$:
   (a) $\mathsf{lcm}(x, y, z)$ is to hold iff z is the least common multiple of x and y;
   (b) $\mathsf{cong}(x, y, z)$ is to hold iff x is congruent to y mod z;
   (c) $\mathsf{sum}(x)$ is to hold iff x can be expressed as $1 + 2 + 3 + \cdots + n$ for some n;
   (d) $\mathsf{sq2sum}(x)$ is to hold iff x can be written as the sum of two squares.

   Important Note: For a first-order formula over $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ to express an n-ary relation on $\mathbb{N}$, it must contain n *free* variables.
9. Translate the following English statements about $\mathbb{N}$ into first-order sentences over $\mathcal{L} = \{+, \cdot, <, 0, 1\}$:
   (a) (**Lagrange**) Every natural number is the sum of four squares.
   (b) (**Bertrand**) For every positive integer n, there is a prime number between n and $2n$.
   (c) (**Pythagorean Triples**) There are infinitely many Pythagorean triples (a, b, c) (i.e., length of sides of some right triangle, with c being the length of the hypotenuse).
   (d) (**Mordell**) For any $k \neq 0$, *Bachet's equation* $y^2 = x^3 + k$ has only finitely many solutions.
10. Let $\mathcal{L} = \{r\}$ be the language of directed graphs. Write a first-order formula that expresses the following relations on the vertices of an arbitrary directed graph $\mathbf{G}$:
    (a) $T(x)$ that holds iff there exists a proper directed triangle (all three sides different) with x as one of its three vertices.
    (b) $P_n(x, y)$ that holds iff there exists a path of length at most n from the vertex x to the vertex y (n fixed);