HOMEWORK 5 SOLUTIONS - MATH 300
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Problem 1 *Consider the directed graph* $\mathbf{G}$ *given by the binary relation "is less than" on the set of numbers* $G = \{0, 1, 2, 3, 4\}$.

(a) *Express* $\mathbf{G}$ *as a set of ordered pairs.*

(b) *Give a table for* $\mathbf{G}$.

(c) *Draw* $\mathbf{G}$.

Solution: Let us denote by $r^{\mathbf{G}}$ the interpretation "is less than" of the binary relation symbol r in the set $G = \{0, 1, 2, 3, 4\}$. Then we have:

(a) $r^{\mathbf{G}} = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.

(b) The table for $r^{\mathbf{G}}$ is given below:

$r^{\mathbf{G}}$	0	1	2	3	4
0	0	1	1	1	1
1	0	0	1	1	1
2	0	0	0	1	1
3	0	0	0	0	1
4	0	0	0	0	0

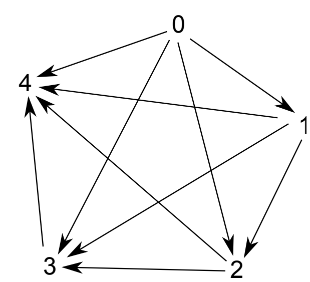

Figure 1: The graph of $r^{\mathbf{G}}$.

(c) The graph shown in Figure 1 is the graph of $r^{\mathbf{G}}$.

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Problem 2 *Consider the directed graph* $\mathbf{G}$ *given by the binary relation "divides" on the set of numbers* $G = \{0, 1, 2, \ldots, 10\}$.

(a) *Express* $\mathbf{G}$ *as a set of ordered pairs.*

(b) *Give a table for* $\mathbf{G}$.

(c) *Draw* $\mathbf{G}$.

Solution: Let us denote by $r^{\mathbf{G}}$ the interpretation "divides" of the binary relation symbol r in the set $G = \{0, 1, 2, \ldots, 10\}$. Then we have:

(a)

$$\begin{aligned} r^{\mathbf{G}} = \; & \{(0, 0), (1, 0), (1, 1), (1, 2), \ldots, (1, 10), (2, 0), (2, 2), (2, 4), (2, 6), (2, 8), (2, 10), \\ & (3, 0), (3, 3), (3, 6), (3, 9), (4, 0), (4, 4), (4, 8), (5, 0), (5, 0), (5, 10), (6, 0), (6, 6), \\ & (7, 0), (7, 7), (8, 0), (8, 8), (9, 0), (9, 9), (10, 0), (10, 10)\}. \end{aligned}$$

(b) The table for $r^{\mathbf{G}}$ is given below:

$r^{\mathbf{G}}$	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
2	1	0	1	0	1	0	1	0	1	0	1
3	1	0	0	1	0	0	1	0	0	1	0
4	1	0	0	0	1	0	0	0	1	0	0
5	1	0	0	0	0	1	0	0	0	0	1
6	1	0	0	0	0	0	1	0	0	0	0
7	1	0	0	0	0	0	0	1	0	0	0
8	1	0	0	0	0	0	0	0	1	0	0
9	1	0	0	0	0	0	0	0	0	1	0
10	1	0	0	0	0	0	0	0	0	0	1

(c) The graph shown in Figure 2 is the graph of $r^{\mathbf{G}}$ except that it does not show the loops that are present on every vertex.

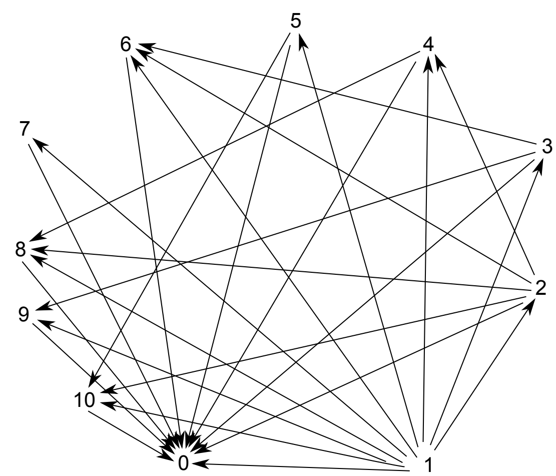

Figure 2: The graph of $r^{\mathbf{G}}$.

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Problem 3 *For the following formulas, state clearly whether they are atomic or non-atomic, create the (formula) syntax tree and apply carefully (step-by-step) the recursive procedure for finding all subformulas:*

(a) $\forall y(r_2 f_1 xy)$

(b) $r_3 f_3 yzzy f_1 z$

(c) $\exists z((r_1 z) \leftrightarrow (r_1 f_2 yz))$

(d) $\forall z((r_2 zx) \to \exists y((r_2 f_1 xy) \to ((r_1 z) \to \forall x(r_1 f_1 x))))$

Solution: We answer all questions for each formula listed in turn:

(a) The formula $\forall y(r_2 f_1 xy)$ is not atomic, since it uses a quantifier; Its syntax tree is

$$\begin{tikzcd} \forall y(r_2 f_1 xy) \\ r_2 f_1 xy \arrow[u] \end{tikzcd}$$

and the subformulas starting with the root and proceeding recursively (each step indented to the right) are

$$\forall y(r_2 f_1 xy)$$
$$\qquad\qquad r_2 f_1 xy \quad \text{(atomic)}$$

(b) The formula $r_3 f_3 yzzyf_1 z$ is atomic, since it consists of a ternary relation symbol followed by three terms; Its syntax tree consists of the single leaf

$$r_3 f_3 yzzyf_1 z$$

and its only subformula is itself because of atomicity.

(c) The formula $\exists z((r_1 z) \leftrightarrow (r_1 f_2 yz))$ is not atomic, since it uses a connective and a quantifier; Its syntax tree is

$$\begin{tikzcd}
& \exists z((r_1 z) \leftrightarrow (r_1 f_2 yz)) & \\
& (r_1 z) \leftrightarrow (r_1 f_2 yz) \arrow[u] & \\
r_1 z \arrow[ur] & & r_1 f_2 yz \arrow[ul]
\end{tikzcd}$$

and the subformulas starting with the root and proceeding recursively (each step indented to the right) are

$$\exists z((r_1 z) \leftrightarrow (r_1 f_2 yz))$$
$$\qquad\qquad (r_1 z) \leftrightarrow (r_1 f_2 yz)$$
$$\qquad\qquad\qquad\qquad r_1 z \quad \text{(atomic)}$$
$$\qquad\qquad\qquad\qquad r_1 f_2 yz \quad \text{(atomic)}$$

(d) The formula $\forall z((r_2 zx) \to \exists y((r_2 f_1 xy) \to ((r_1 z) \to \forall x(r_1 f_1 x))))$ is not atomic, since it uses connectives and quantifiers; Its syntax tree is

$$\begin{tikzcd}
& \forall z((r_2 zx) \to \exists y((r_2 f_1 xy) \to ((r_1 z) \to \forall x(r_1 f_1 x)))) & & & \\
& (r_2 zx) \to \exists y((r_2 f_1 xy) \to ((r_1 z) \to \forall x(r_1 f_1 x))) \arrow[u] & & & \\
r_2 zx \arrow[ur] & & \exists y((r_2 f_1 xy) \to ((r_1 z) \to \forall x(r_1 f_1 x))) \arrow[ul] & & \\
& & (r_2 f_1 xy) \to ((r_1 z) \to \forall x(r_1 f_1 x)) \arrow[u] & & \\
& r_2 f_1 xy \arrow[ur] & & (r_1 z) \to \forall x(r_1 f_1 x) \arrow[ul] & \\
& & r_1 z \arrow[ur] & & \forall x(r_1 f_1 x) \arrow[ul] \\
& & & & r_1 f_1 x \arrow[u]
\end{tikzcd}$$

and the subformulas starting with the root and proceeding recursively (each step indented to the right) are

$$\begin{array}{l}
\forall z((r_2zx) \to \exists y((r_2f_1xy) \to ((r_1z) \to \forall x(r_1f_1x)))) \\
\qquad (r_2zx) \to \exists y((r_2f_1xy) \to ((r_1z) \to \forall x(r_1f_1x))) \\
\qquad\quad r_2zx \\
\qquad\quad \exists y((r_2f_1xy) \to ((r_1z) \to \forall x(r_1f_1x))) \\
\qquad\qquad (r_2f_1xy) \to ((r_1z) \to \forall x(r_1f_1x)) \\
\qquad\qquad\quad r_2f_1xy \\
\qquad\qquad\quad (r_1z) \to \forall x(r_1f_1x) \\
\qquad\qquad\qquad r_1z \\
\qquad\qquad\qquad \forall x(r_1f_1x) \\
\qquad\qquad\qquad\quad r_1f_1x
\end{array}$$

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Problem 4 *Do the following for each first-order formula below:* (a) *Indicate, for each variable involved, its bound and its free occurrences.* (b) *Tell clearly the scope of each quantifier.* (c) *For every bound occurrence of a variable, indicate which occurrence of a quantifier binds it.*

(i) $\forall y \exists y((r_2xz) \lor \exists z(r_2xz))$

(ii) $\forall x(\forall z(((r_1y) \land \neg(r_2zy)) \lor \exists y(r_3yxz)) \land (r_2yz))$

(iii) $(\exists y(r_3xyz)) \to \exists x((\forall x(r_3zyy)) \to (r_1f_3xyx))$

Solution:

(i) Consider the formula $\forall y \exists y((r_2xz) \lor \exists z(r_2xz))$.

(a) There are two occurrences of x, both free. There are two occurrences of y, both bound. There are three occurrences of z; the first is free and the second and third are bound.

(b) Following table summarizes quantifiers and their scopes:

Quantifier	Scope
$\forall y$	$\forall y \exists y((r_2xz) \lor \exists z(r_2xz))$
$\exists y$	$\exists y((r_2xz) \lor \exists z(r_2xz))$
$\exists z$	$\exists z(r_2xz)$

(c) The first bound occurrence of y is bound by $\forall y$, the second is bound by $\exists y$. The bound occurrences of z are bound by $\exists z$.

(ii) Consider the formula $\forall x(\forall z(((r_1y) \land \neg(r_2zy)) \lor \exists y(r_3yxz)) \land (r_2yz))$.

(a) There are two occurrences of x, both bound. There are five occurrences of y; the first, second and fifth are free whereas the third and fourth are bound. There are four occurrences of z; the first three are bound and the last is free.

(b) Following table summarizes quantifiers and their scopes:

Quantifier	Scope
$\forall x$	$\forall x(\forall z(((r_1y) \land \neg(r_2zy)) \lor \exists y(r_3yxz)) \land (r_2yz))$
$\forall z$	$\forall z(((r_1y) \land \neg(r_2zy)) \lor \exists y(r_3yxz))$
$\exists y$	$\exists y(r_3yxz)$

(c) The bound occurrences of x are bound by $\forall x$, the bound occurrences of y by $\exists y$ and the bound occurrences of z by $\forall z$.

(iii) Consider the formula $(\exists y(r_3xyz)) \to \exists x((\forall x(r_3zyy)) \to (r_1f_3xyx))$.

(a) There are five occurrences of x; the first is free and the last four are bound. There are five occurrences of y; the first two are bound and the last three are free. There are two occurrences of z, both free.

(b) Following table summarizes quantifiers and their scopes:

Quantifier	Scope
$\exists y$	$\exists y(r_3xyz)$
$\exists x$	$\exists x((\forall x(r_3zyy)) \to (r_1f_3xyx))$
$\forall x$	$\forall x(r_3zyy)$

(c) The second, fourth and fifth occurrences of x are bound by $\exists x$, whereas the third is bound by $\forall x$. The bound occurrences of y are bound by $\exists y$.

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Problem 5 *This exercise involves formulas over the language* $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ *of the natural numbers* $\mathbb{N}$. *For each string, state (with a very brief explanation, when appropriate) whether it is a formula. If yes, state clearly whether it is atomic, whether it is a sentence, build its (formula) syntax tree and find all its subformulas.*

(a) $1 \cdot 1 \approx 1 + 1$

(b) $(1 + 1 < 1) \approx 0$

(c) $(x \cdot (x \approx y) \cdot z) \leftrightarrow (1 < (x + (y \cdot z)))$

(d) $\forall x(x + x \approx (1 + 1) \cdot x)$

(e) $\forall z(((z \approx x) \wedge (y < x)) \vee \exists y \exists z(z \approx (z \cdot y)))$

Solution:

(a) The string $1 \cdot 1 \approx 1 + 1$ represents a valid formula; it is atomic of the equational kind and a sentence, since it does not contain any (free) variables. Therefore, its syntax tree consists of the unique leaf

$$1 \cdot 1 \approx 1 + 1$$

and its only subformula is itself.

(b) The string $(1 + 1 < 1) \approx 0$ does not represent a valid formula. The outermost $\approx$ is not connecting two valid terms of the given language.

(c) The string $(x \cdot (x \approx y) \cdot z) \leftrightarrow (1 < (x + (y \cdot z)))$ does not represent a valid formula either. The outermost $\leftrightarrow$ must connect two valid $\mathcal{L}$-formulas. However, the expression $x \cdot (x \approx y) \cdot z$ is not a formula.

(d) The string $\forall x(x + x \approx (1 + 1) \cdot x)$ represents a valid formula; it is not atomic since it uses a quantifier, but it is a sentence, since it does not contain any free variables. Its syntax tree is given below

$$\begin{array}{c} \forall x(x + x \approx (1 + 1) \cdot x) \\ \uparrow \\ x + x \approx (1 + 1) \cdot x \end{array}$$

and its subformulas are the sentence $\forall x(x + x \approx (1 + 1) \cdot x)$ itself and the atomic formula $x + x \approx (1 + 1) \cdot x$.

(e) The string $\forall z(((z \approx x) \wedge (y < x)) \vee \exists y \exists z(z \approx (z \cdot y)))$ represents a valid formula. It is not atomic, since it uses connectives and a quantifiers; it is not a sentence, since both occurrences of x and the first occurrence of y are free. Its syntax tree is

$$
\begin{tikzcd}[column sep=small]
 & & & \forall z(((z \approx x) \wedge (y < x)) \vee \exists y \exists z(z \approx (z \cdot y))) & \\
 & & & ((z \approx x) \wedge (y < x)) \vee \exists y \exists z(z \approx (z \cdot y)) \arrow[u] & \\
 & (z \approx x) \wedge (y < x) \arrow[urr] & & & \exists y \exists z(z \approx (z \cdot y)) \arrow[ul] \\
 z \approx x \arrow[ur] & & y < x \arrow[ul] & & \exists z(z \approx (z \cdot y)) \arrow[u] \\
 & & & & z \approx (z \cdot y) \arrow[u]
\end{tikzcd}
$$

and the subformulas starting with the root and proceeding recursively (each step indented to the right) are

$$
\begin{array}{l}
\forall z(((z \approx x) \wedge (y < x)) \vee \exists y \exists z(z \approx (z \cdot y))) \\
\qquad ((z \approx x) \wedge (y < x)) \vee \exists y \exists z(z \approx (z \cdot y)) \\
\qquad\qquad ((z \approx x) \wedge (y < x)) \\
\qquad\qquad\qquad z \approx x \\
\qquad\qquad\qquad y < x \\
\qquad\qquad \exists y \exists z(z \approx (z \cdot y)) \\
\qquad\qquad\qquad \exists z(z \approx (z \cdot y)) \\
\qquad\qquad\qquad\qquad z \approx (z \cdot y)
\end{array}
$$

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Problem 6 *This exercise involves formulas over the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ of the natural numbers $\mathbb{N}$. Do the following for each formula below:* (a) *Indicate, for each variable involved, its bound and its free occurrences.* (b) *Tell clearly the scope of each quantifier.* (c) *For every bound occurrence of a variable, indicate which occurrence of a quantifier binds it.*

(i) $\neg(\forall x(x \approx y) \leftrightarrow \exists x((y \approx x) \wedge (x < z)))$

(ii) $\forall y((z < (x \cdot y)) \rightarrow ((z \approx z) \wedge \forall x(y < x)))$

(iii) $(\forall x \neg(\exists x(x < y) \vee \neg((x < y) \wedge \exists x(y < x)))) \wedge (z \approx z)$

Solution:

(i) Consider the formula $\neg(\forall x(x \approx y) \leftrightarrow \exists x((y \approx x) \wedge (x < z)))$.

  (a) There are five occurrences of x, all of them bound. There are two occurrences of y and one occurrence of z, all three occurrences are free.

  (b) Following table summarizes quantifiers and their scopes:

Quantifier	Scope
$\forall x$	$\forall x(x \approx y)$
$\exists x$	$\exists x((y \approx x) \wedge (x < z))$

(c) The first two bound occurrences of x are bound by $\forall x$, the last three are bound by $\exists x$.

(ii) Consider the formula $\forall y((z < (x \cdot y)) \rightarrow ((z \approx z) \wedge \forall x(y < x)))$.

(a) There are three occurrences of x, the first free and the last two bound. There are three occurrences of y, all bound. There are also three occurrences of z, all of them free.

(b) Following table summarizes quantifiers and their scopes:

Quantifier	Scope
$\forall y$	$\forall y((z < (x \cdot y)) \rightarrow ((z \approx z) \wedge \forall x(y < x)))$
$\forall x$	$\forall x(y < x)$

(c) The last two occurrences of x are bound by $\forall x$ and all three occurrences of y are bound by $\forall y$.

(iii) Consider the formula $(\forall x \neg(\exists x(x < y) \vee \neg((x < y) \wedge \exists x(y < x)))) \wedge (z \approx z)$.

(a) There are six occurrences of x, all of them bound. There are three occurrences of y, all free, and two occurrences of z, both of them free.

(b) Following table summarizes quantifiers and their scopes:

Quantifier	Scope
$\forall x$	$\forall x \neg(\exists x(x < y) \vee \neg((x < y) \wedge \exists x(y < x)))$
$\exists x$	$\exists x(x < y)$
$\exists x$	$\exists x(y < x)$

(c) The first bound occurrence of x is bound by $\forall x$, the second and third occurrences are bound by the first $\exists x$, the fourth occurrence is bound by the $\forall x$, while the last two occurrences of x are bound by the second $\exists x$.

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Problem 7 *Translate the following first-order sentences over the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ into English:*

(a) $\forall x(\exists y(x \approx y + y + 1) \rightarrow \exists y(x + 1 \approx y + y))$

(b) $\forall x((\exists y(x \approx 3 \cdot y) \wedge \exists y(x \approx 5 \cdot y)) \rightarrow \exists y(x \approx 15 \cdot y))$

(c) $\forall x(x \approx 0 \vee \forall y((\neg(y < x) \wedge \neg(y \approx x)) \rightarrow \neg\exists z(x \approx z \cdot y)))$

(d) $$\exists x \exists y(\neg(x \approx y) \wedge \exists w(x \approx 7 \cdot w) \wedge \exists w(y \approx 7 \cdot w) \wedge \\ \forall z((\exists w(x \approx z \cdot w) \wedge \exists w(y \approx z \cdot w)) \rightarrow ((z < 7) \vee (z \approx 7))))$$

Important Note: *This problem is not asking you to simply* transliterate *first-order into English. For example, "there exists x, such that for all y, x is less than y or x is equal to y" is the transliteration of $\exists x \forall y((x < y) \vee (x \approx y))$, but* **not** *its English* translation. *Rather, an English translation, i.e., a sentence that would naturally express in English what this sentence means, is "there exists a smallest element x". Thus, even though transliteration is an almost mechanical phonetic process, translation requires some understanding of the meaning of the sentence in the intended interpretation structure(s).*

Solution:

(a) The sentence $\forall x(\exists y(x \approx y + y + 1) \rightarrow \exists y(x + 1 \approx y + y))$ formally expresses the statement that, For every natural number x, if x is odd, then $x+1$ is even.

(b) The sentence $\forall x((\exists y(x \approx 3 \cdot y) \wedge \exists y(x \approx 5 \cdot y)) \rightarrow \exists y(x \approx 15 \cdot y))$ formally expresses the statement that,

For every natural number x, if $3 \backslash x$ and $5 \backslash x$, then $15 \backslash x$.

(c) The sentence $\forall x(x \approx 0 \vee \forall y((\neg(y < x) \wedge \neg(y \approx x)) \rightarrow \neg \exists z(x \approx z \cdot y)))$ formally expresses the statement that,

For every natural number x, either $x = 0$ or, if a number y divides x, then $y \leq x$.

(d) The sentence $\exists x \exists y(\neg(x \approx y) \wedge \exists w(x \approx 7 \cdot w) \wedge \exists w(y \approx 7 \cdot w) \wedge$
$\forall z((\exists w(x \approx z \cdot w) \wedge \exists w(y \approx z \cdot w)) \rightarrow ((z < 7) \vee (z \approx 7))))$ formally expresses the statement that,

There exist two difference natural numbers whose greatest common divisor is 7.

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Problem 8 *Write first-order formulas over $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ that express the following relations on $\mathbb{N}$:*

(a) $\mathsf{lcm}(x, y, z)$ *is to hold iff z is the least common multiple of x and y;*

(b) $\mathsf{cong}(x, y, z)$ *is to hold iff x is congruent to y mod z;*

(c) $\mathsf{sum}(x)$ *is to hold iff x can be expressed as $1 + 2 + 3 + \cdots + n$ for some n;*

(d) $\mathsf{sq2sum}(x)$ *is to hold iff x can be written as the sum of two squares.*

Important Note: *For a first-order formula over $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ to express an n-ary relation on $\mathbb{N}$, it must contain n* free *variables.*

Solution:

(a) The ternary relation $\mathsf{lcm}(x, y, z)$ on $\mathbb{N}$ that is to hold iff z is the least common multiple of x and y is formally expressed by the following first-order formula in the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$:

$$\exists w(z \approx w \cdot x) \wedge \exists w(z \approx w \cdot y) \wedge \forall u((\exists w(u \approx w \cdot x) \wedge \exists w(u \approx w \cdot y)) \rightarrow ((z < u) \vee (z \approx u))).$$

(b) The ternary relation $\mathsf{cong}(x, y, z)$ on $\mathbb{N}$ that is to hold iff x is congruent to y mod z is formally expressed by the following first-order formula in the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$:

$$\exists w((x \approx y + z \cdot w) \vee (y \approx x + z \cdot w)).$$

(c) Recall that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Therefore the unary relation $\mathsf{sum}(x)$ on $\mathbb{N}$ that is to hold iff x can be expressed as $1+2+3+\cdots+n$ for some n, i.e., iff $x = \frac{n(n+1)}{2}$ for some n, is formally expressed by the following first-order formula in the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$:

$$\boxed{\exists y(x + x \approx y \cdot (y+1)).}$$

(d) The unary relation $\mathsf{sq2sum}(x)$ on $\mathbb{N}$ that is to hold iff x can be written as the sum of two squares is formally expressed by the following first-order formula in the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$:

$$\boxed{\exists y \exists z(x \approx (y \cdot y) + (z \cdot z)).}$$

∎

Problem 9 *Translate the following English statements about $\mathbb{N}$ into first-order sentences over $\mathcal{L} = \{+, \cdot, <, 0, 1\}$:*

(a) (**Lagrange**) *Every natural number is the sum of four squares.*

(b) (**Bertrand**) *For every positive integer n, there is a prime number between n and $2n$.*

(c) (**Pythagorean Triples**) *There are infinitely many Pythagorean triples (a, b, c) (i.e., length of sides of some right triangle, with c being the length of the hypotenuse).*

(d) (**Mordell**) *For any $k \neq 0$,* Bachet's equation $y^2 = x^3 + k$ *has only finitely many solutions.*

Solution:

(a) The statement "Every natural number is the sum of four squares" is formally expressed by the following first-order sentence in the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$:

$$\boxed{\forall x \exists y \exists z \exists w \exists u(x = (y \cdot y) + (z \cdot z) + (w \cdot w) + (u \cdot u)).}$$

(b) The statement "For every positive integer n, there is a prime number between n and $2n$" is formally expressed by the following first-order sentence in the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$:

$$\boxed{\forall x((x > 0) \to \exists y((x < y) \wedge (y < x + x) \wedge (1 < y) \wedge \forall z(\exists w(y \approx w \cdot z) \to ((z \approx 1) \vee (z \approx y))))).}$$

(c) The statement "There are infinitely many Pythagorean triples (x, y, z)" is formally expressed by the following first-order sentence in the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$:

$$\boxed{\forall w \exists x \exists y \exists z((w < z) \wedge (z \cdot z \approx (x \cdot x) + (y \cdot y)).}$$

(d) The statement "For any $k \neq 0$, *Bachet's equation* $y^2 = x^3 + k$ has only finitely many solutions" is formally expressed by the following first-order sentence in the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$:

$$\boxed{\forall z((z > 0) \to \exists w \forall y((y \cdot y \approx ((x \cdot x) \cdot x) + z) \to (y < w))).}$$

∎

Problem 10 *Let $\mathcal{L} = \{r\}$ be the language of directed graphs. Write a first-order formula that expresses the following relations on the vertices of an arbitrary directed graph* $\mathbf{G}$*:*

(a) *$T(x)$ that holds iff there exists a proper directed triangle (all three sides different) with x as one of its three vertices.*

(b) *$P_n(x, y)$ that holds iff there exists a path of length at most n from the vertex x to the vertex y (n fixed);*

Solution:

(a) In the formula over $\mathcal{L}$ that expresses $T(x)$ that holds iff there exists a proper directed triangle with x as one of its three vertices, we try to capture the following diagram:

$$\begin{tikzcd}
 & y \arrow[dd] \\
x \arrow[ur] & \\
 & z \arrow[ul]
\end{tikzcd}$$

$$\exists y \exists z((rxy) \wedge (ryz) \wedge (rzx) \wedge \neg(x \approx y) \wedge \neg(x \approx z) \wedge \neg(y \approx z)).$$

(b) Note that there exists a path of length at most n from a vertex x to a vertex y iff either $x = y$ or when there exists a path of length i, for some i, with $1 \leq i \leq n$, consisting of not necessarily distinct vertices; This last statement may be expresses by

$$x \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \qquad\qquad \longrightarrow x_{i-1} \longrightarrow y$$

$$\exists x_1 \cdots \exists x_{i-1}((rxx_1) \wedge (rx_1x_2) \wedge \cdots \wedge (rx_{i-1}y)).$$

Therefore, a formula over $\mathcal{L}$ that expresses $P_n(x, y)$ that holds iff there exists a path of length at most n from the vertex x to the vertex y is the following:

$$(x \approx y) \vee \bigvee_{i=1}^{n} \left[\exists x_1 \cdots \exists x_{i-1} \left((rxx_1) \wedge \bigwedge_{j=1}^{i-2} (rx_jx_{j+1}) \wedge (rx_{i-1}y) \right) \right].$$

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