HOMEWORK 6 - MATH 300 Due: Wednesday, April 17

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Read each problem **very carefully** and try to understand it well before starting to solve it. It is necessary to show **all** your work. Correct answers without explanations are worth 0 points. Write your own solutions and be neat!! **Take pride in your work!! Do not hand in scratchy doodles.**

1. Consider the first-order language $\mathcal{L} = \{f, r\}$, where f is a ternary function symbol and r a binary relation symbol. Consider also the $\mathcal{L}$-structure $\mathbf{S} = (S, I)$, where $S = \{0, 1\}$ and I consists of the following interpretations of the function symbol f and the relation symbol r:

			$f^{\mathbf{S}}$
1	1	1	0
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	1

$r^{\mathbf{S}}$	0	1
0	0	1
1	0	0

   Determine the truth of the following first-order sentences in $\mathbf{S}$:

   (a) $\forall x(rxfxxx)$

   (b) $\forall x\exists y(rfxyxfxxx)$

2. Consider the first-order language $\mathcal{L} = \{f, r\}$, where f is a ternary function symbol and r a binary relation symbol. Consider also the $\mathcal{L}$-structure $\mathbf{S} = (S, I)$, where $S = \{0, 1, 2\}$ and I consists of the following interpretations of the function symbol f and the relation symbol r:

$$f^{\mathbf{S}}(a, b, c) = (a - b + c) \bmod 3$$

$r^{\mathbf{S}}$	0	1	2
0	1	0	0
1	1	1	1
2	1	0	1

   Determine the truth of the following first-order sentences in $\mathbf{S}$:

   (a) $\forall x\exists y((rfxyxy) \to (ryfxyx))$

   (b) $\forall x\exists y\forall z(rxfzyx)$

3. Consider the language $\mathcal{L} = \{+, \cdot, <\}$, where $+, \cdot$ are binary function symbols and $<$ is a binary relation symbol. Let $x \leq y$ be an abbreviation for $(x < y) \vee (x \approx y)$. Use the skeleton method to determine whether each of the following $\mathcal{L}$-sentences has a one-element model. If yes, then exhibit that one-element model.

   (a) $\forall x(\exists y(x \cdot y < x) \to \exists z((y \cdot z < x) \wedge \neg(y \cdot x < z))) \to \exists x\forall y(x \cdot y < y)$

   (b) $\forall x(\exists y(x \cdot y \leq x) \to \neg\exists y\forall z((y + z < x \cdot y) \wedge \neg(y \cdot x \leq z))) \to \exists x\forall y(x + y < x \cdot y)$

4. To show that two $\mathcal{L}$-sentences F and G are equivalent, we must either use some of our fundamental equivalences from first-order logic to transform one to the other, or we have to prove that they are true in exactly the same structures, i.e., that for every structure $\mathbf{S}$, $\mathbf{S} \models F$ iff $\mathbf{S} \models G$. On the other hand, to show that they are not equivalent, it suffices to find a single structure $\mathbf{S}$ in which one of the two sentences is true and the other is false.

   Determine if the following pairs of $\mathcal{L}$-sentences are equivalent, where $\mathcal{L} = \{f\}$, with f a unary function symbol:

(a) $\forall x((ffx \approx x) \wedge (fffx \approx x))$ and $\forall x(fx \approx x)$;

(b) $\forall x \forall y((fx \approx fy) \rightarrow (x \approx y))$ and $\forall y \exists x(fx \approx y)$.

5. To show that two $\mathcal{L}$-formulas $F(x)$ and $G(x)$, with a free variable x are equivalent, we must either use some of our fundamental equivalences from first-order logic to transform one to the other, or we have to prove that they define exactly the same unary relations in all $\mathcal{L}$-structures, i.e., that for every structure $\mathbf{S}$ and every $a \in S$, $F^{\mathbf{S}}(a)$ holds iff $G^{\mathbf{S}}(a)$ holds. On the other hand, to show that they are not equivalent, it suffices to find a single structure $\mathbf{S}$ and a single element $a \in S$, such that one of $F^{\mathbf{S}}(a), G^{\mathbf{S}}(a)$ is true and the other is false.

   Determine if the following pairs of formulas are equivalent:

   (a) $\forall y(rxy)$ and $\exists y(rxy)$, where $\mathcal{L} = \{r\}$, r a binary relation symbol;

   (b) $\exists y(r_1 fy \wedge r_2 y \wedge (x \approx fy))$ and $\exists y \exists z(r_1 y \wedge r_2 z \wedge (x \approx fy) \wedge (x \approx fz))$, where $\mathcal{L} = \{f, r_1, r_2\}$, with f a unary function symbol and r_1, r_2 unary relation symbols.

6. Put the following formulas in the language of graphs into prenex normal form:

   (a) $(\forall y(\neg(y \approx z) \vee \forall y(y \approx z))) \vee (ryz)$

   (b) $(z \approx x) \vee ((\forall x(\neg \forall z(rxz))) \rightarrow (y \approx z))$

   (c) $(x \approx y) \rightarrow \exists y(((y \approx z) \rightarrow (\exists z(y \approx z))) \wedge (y \approx z))$

7. Find a counterexample for the following arguments:

   (a) (Here $\mathcal{L} = \{r_1, r_2, r_3\}$, with r_1, r_2, r_3 unary relation symbols.)

$$\begin{array}{l} \forall x(r_1 x \rightarrow (r_2 x \rightarrow r_3 x)) \\ \therefore\ \forall x((r_1 x \rightarrow r_2 x) \rightarrow r_3 x) \end{array}$$

   (b) (Here $\mathcal{L} = \{r\}$, with r a binary relation symbol.)

$$\begin{array}{l} \forall x \exists y(rxy) \\ \forall y \exists x(rxy) \\ \therefore\ \forall x \forall y(\neg(x \approx y) \rightarrow (rxy)) \end{array}$$

8. Skolemize the following sentences:

   (a) $\forall x \exists y(x < y)$

   (b) $\exists x \forall y(x < y)$

   (c) $\forall x \forall y((rxy) \rightarrow \exists z((rxz) \wedge (rzy)))$