Communication Complexity

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The Communication Game

- The game involves two players, Alice and Bob.
- They both know a function

$$f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}.$$

- Only Alice knows the first half $x \in \{0,1\}^n$ of the input;
- Only Bob knows the second half $y \in \{0,1\}^n$ of the input.
- Their goal is to compute the value $f(x,y)$.
- They have agreed to operate (and communicate) according to a specified protocol Π to compute $f(x,y)$.

Communication Protocol

- A t-**round protocol** Π is a sequence of t functions

$$P_1, \ldots, P_t : \{0,1\}^* \to \{0,1\}^*.$$

- An **execution of** Π on inputs $x, y \in \{0,1\}^n$ goes as follows:

Alice computes $p_1 = P_1(x)$ and sends p_1 to Bob;
Bob computes $p_2 = P_2(y, p_1)$ and sends p_2 to Alice;
$\vdots$
odd i Alice computes $p_i = P_i(x, p_1, \ldots, p_{i-1})$ and sends p_i to Bob;
even i Bob computes $p_i = P_i(y, p_1, \ldots, p_{i-1})$ and sends p_i to Alice;
$\vdots$

- Π is a **protocol for** f if, for all $x, y \in \{0,1\}^n$, $p_t = f(x, y)$, i.e., the last bit communicated is the correct value of f, for all inputs (x, y).

Communication Complexity

- Communication complexity diverges dramatically from ordinary notions of complexity.
  - It ignores the complexities of computing the P_i's;
  - It only focuses on the number of bits that are communicated.
- The **communication complexity of a protocol** Π is the maximum number of bits communicated over all inputs:

$$C(\Pi) = \max_{x,y \in \{0,1\}^n} \{|p_1| + |p_2| + \cdots + |p_t|\}.$$

- The **communication complexity of a function** f is the minimum communication complexity over all protocols Π for f:

$$C(f) = \min_{\Pi \text{ for } f} C(\Pi).$$

A Trivial Upper Bound

- For every $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$,

$$C(f) \leq n+1.$$

- Here is a protocol Π^*:
  - $p_1 = x$;
  - $p_2 = f(x,y)$.

  We have $C(\Pi^*) = n+1$.

  So we get

$$C(f) = \min_{\Pi \text{ for } f} C(\Pi) \leq n+1.$$

The Parity Function

- Suppose
$$f(x,y) = \bigoplus_{i=1}^{n} x_i \oplus \bigoplus_{i=1}^{n} y_i.$$
- If Π is a protocol for f, then $C(\Pi) \geq 2$.

  Follows from the fact that f depends nontrivially on both x and y.
- Now consider the following protocol Π^*:
  - $p_1 = \bigoplus_{i=1}^{n} x_i$;
  - $p_2 = p_1 \oplus \bigoplus_{i=1}^{n} y_i$.

  We have $C(\Pi^*) = 2$.

  It follows that $C(f) = \min_{\Pi \text{ for } f} C(\Pi) \leq 2$.
- Therefore, $C(f) = 2$.

Matching a Bit

- We consider the function of 10 bits

$$f(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, s_0, s_1) = 1 \quad \text{iff} \quad a_{s_0+2s_1} = b_{s_0+2s_1}.$$

  - Alice knows a_0, a_1, a_2, a_3, s_0;
  - Bob knows b_0, b_1, b_2, b_3, s_1.
- A protocol Π for f is as follows:
  - $p_1 = s_0$;
  - $p_2 = \langle s_1, b_{s_0+2s_1} \rangle$;
  - $p_3 = \begin{cases} 1, & \text{if } a_{s_0+2s_1} = b_{s_0+2s_1} \\ 0, & \text{otherwise} \end{cases}$
- We have $C(\Pi) = 4$.

Fooling Sets

The Fooling Sets Lemma

Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be a function and Π a protocol for f. Suppose that for $x, y \in \{0,1\}^n$, with $x \neq y$, $\Pi(x,x) = \Pi(y,y)$. Then,

$$f(x,x) = f(x,y) = f(y,x) = f(y,y).$$

- By hypothesis, $f(x,x) = f(y,y)$.
  By symmetry, it suffices to show that $f(x,x) = f(x,y)$.
  We show by induction on i that p_i (on (x,x)) and q_i (on (x,y)) agree.
  - $p_1 = P_1(x) = q_1$;
  - Assume $p_j = q_j$, for all $j < i$.
  - Then, if i is odd, the conclusion is obvious.
    If i is even, $p_i = P_i(x, p_1, \ldots, p_{i-1}) = P_i(y, q_1, \ldots, q_{i-1}) = q_i$, where the middle equation follows from $\Pi(x,x) = \Pi(y,y)$.

  For $i = t$, we get the conclusion.

Equality

- Consider the function $\mathrm{EQ} : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, defined by

$$\mathrm{EQ}(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases}$$

Theorem

$C(\mathrm{EQ}) \geq n$.

- Suppose there exists a protocol Π for EQ, such that $C(\Pi) \leq n-1$.
  Then, there exist at most 2^{n-1} distinct communication patterns.
  But there are at least 2^n distinct inputs of the form (x,x).
  Hence, there exist $x, y \in \{0,1\}^n$, $x \neq y$, such that $\Pi(x,x) = \Pi(y,y)$.
  By the Fooling Sets Lemma, $\mathrm{EQ}(x,x) = \mathrm{EQ}(x,y)$, a contradiction.

Matrix of f, Rectangles and Monochromatic Tilings

- Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be a function.
- The **matrix of** f, $M(f)$, is a $2^n \times 2^n$ matrix, such that, for all $x, y \in \{0,1\}^n$, $M_{x,y} = f(x,y)$.
- A (**combinatorial**) **rectangle in** M is a submatrix of M of the form $A \times B$, for some $A, B \subseteq \{0,1\}^n$.
- The rectangle $A \times B$ is **monochromatic** if $M_{x,y}$ is constant, for all $x \in A$, $y \in B$.
- A **monochromatic tiling of** $M(f)$ is a partition of $M(f)$ into disjoint monochromatic rectangles.
- $\chi(f)$ is the minimum number of rectangles in any monochromatic tiling of $M(f)$.

The Tiling Method

Theorem

For any $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, $C(f) \geq \log_2 \chi(f)$.

- Suppose $C(f) = k$. It suffices to show that $M(f)$ has a monochromatic tiling with at most 2^k rectangles.

  Suppose, in the first round, Alice sends a bit.

  $M(f)$ partitions into $A_0 \times \{0,1\}^n$ and $A_1 \times \{0,1\}^n$, where A_0, A_1 are the subsets of the input for which the bit 0 or 1, respectively, is sent.

  If, in the next round, Bob sends a bit, then each of $A_0 \times \{0,1\}^n$ and $A_1 \times \{0,1\}^n$ is partitioned into two smaller rectangles similarly.

  When the protocol stops (with 2^k rectangles in the partition), all pairs (x, y) in the same rectangle are provided with the same answer.

  So the resulting partition is in monochromatic rectangles.

The Rank Method

- The **rank** of a square matrix M, $\text{rank}(M)$, is the size of the largest subset of rows which are linearly independent.
- A characterization asserts that the rank of a matrix M is the minimum value of ℓ, such that M can be expressed as $M = \sum_{i=1}^{\ell} B_i$, where the B_i's are matrices of rank 1.

Theorem

For any $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, $\chi(f) \geq \text{rank}(M(f))$.

- Every monochromatic rectangle can be viewed as a $2^n \times 2^n$ matrix of rank at most 1.
- Returning to EQ, $\text{rank}(M(\mathrm{EQ})) = 2^n$.

  So $C(\mathrm{EQ}) \geq \log_2 \chi(\mathrm{EQ}) \geq n$.

  So we get an alternative proof of the lower bound.

The Discrepancy Method

- Consider $M(f)$ to be a ± 1 instead of a $0/1$-matrix.
- The **discrepancy of a rectangle** $A \times B$ in a $2^n \times 2^n$ matrix M is

$$\text{Disc}_M(A \times B) = \frac{1}{2^{2n}} \left| \sum_{x \in A, y \in B} M_{x,y} \right|.$$

- The **discrepancy of the matrix** $M(f)$, $\text{Disc}(f)$, is the maximum discrepancy among all rectangles.

Lemma

For any $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, $\chi(f) \geq \frac{1}{\text{Disc}(f)}$.

- Suppose $\chi(f) = k$.
  Then, there exists a monochromatic rectangle with $\geq \frac{2^{2n}}{k}$ entries.
  Its discrepancy is at least $\frac{1}{k}$.
  Putting everything together, $\frac{1}{\text{Disc}(f)} \leq k = \chi(f)$.

Thank you!

- In closing...

Thank you for your Attention!!