Second-Order Logic and Graphs

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Expressions in Second-Order Logic: Example 1

- Consider the second-order expression (in the vocabulary of number theory)

$$\varphi = \exists P \forall x((P(x) \vee P(x+1)) \wedge \neg(P(x) \wedge P(x+1))).$$

- It asserts the existence of a set P such that for all x either $x \in P$ or $x+1 \in P$ but not both.
- φ is satIsfied by $\mathbb{N}$, the standard model of number theory:
  Just take $P^{\mathbb{N}}$ to be the set of even numbers.

Expressions in Second-Order Logic: Example 2

- Consider the sentence

$$\exists P \forall x \forall y (P(x,y) \rightarrow G(x,y))$$

  in the vocabulary of graph theory.
- It asserts the existence of a subgraph of graph G.
- It is a valid sentence, because any graph has at least one subgraph: Namely, itself (not to mention the empty subgraph...).

Expressions in Second-Order Logic

- A **vocabulary** $\Sigma = (\Phi, \Pi, r)$ consists of
  - A set Φ of **function symbols**;
  - A set Π of **relation symbols**;
  - A function $r : \Phi \cup \Pi \to \mathbb{N}$ assigning to each function and each relation symbol in Σ an **arity** (number of arguments).
- An **expression of existential second-order logic** over a vocabulary $\Sigma = (\Phi, \Pi, r)$ is of the form $\exists P \varphi$, where φ is a first-order expression over the vocabulary $\Sigma' = (\Phi, \Pi \cup \{P\}, r)$.
- That is, $P \notin \Pi$ is a new relational symbol of arity $r(P)$.
- Intuitively, expression $\exists P \varphi$ says that there is a relation P such that φ holds.
- A model M appropriate for Σ **satisfies** $\exists P \varphi$ if there is a relation $P^M \subseteq (U^M)^{r(P)}$ such that M, augmented with P^M to comprise a model appropriate for Σ', satisfies φ.

Capturing UnReachability

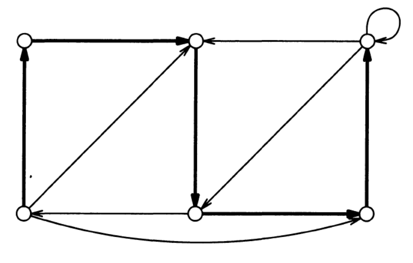
- Our next expression of second-order logic captures graph reachability.
- More precisely, it expresses unreachability, the complement of reachability:

$$\begin{aligned}\varphi(x,y) \quad = \quad & \exists P(\forall u \forall v \forall w(P(u,u) \wedge (G(u,v) \rightarrow P(u,v)) \\ & \wedge((P(u,v) \wedge P(v,w)) \rightarrow P(u,w)) \wedge \neg P(x,y))).\end{aligned}$$

- $\varphi(x,y)$ states that there is a graph P such that:
  - it contains G as a subgraph;
  - it is reflexive and transitive;
  - in this graph there is no edge from x to y.
- It is easy to see that any P that satisfies the first two conditions must contain an edge between any two nodes of G that are reachable (i.e., it must contain the reflexive-transitive closure of G).
- Thus, $\neg P(x,y)$ implies that there is no path from x to y in G.
- $\varphi(x,y)$-Graphs (does a given graph satisfy $\varphi(x,y)$) is precisely the complement of Reachability.

The Problem HAMILTONPATH

- Existential second-order logic can be used to express graph-theoretic properties which, unlike REACHABILITY, have no known polynomial time algorithm.
- Consider the problem HAMILTONPATH:
  Given a graph, is there a path that visits each node exactly once?



- Currently no polynomial time algorithm is known for telling whether a graph has a Hamilton path.

Capturing HamiltonPath

- The following $\psi = \exists P \chi$ describes graphs with a Hamilton path.
- χ will require that P be a linear order on the nodes of G, i.e., a binary relationship isomorphic to $<$ on the nodes of G (which may be taken to be $\{1, 2, \ldots, n\}$), such that consecutive nodes are connected in G.
- χ must require several things:
  - All distinct nodes of G be comparable by P:
    $$\forall x \forall y (P(x, y) \vee P(y, x) \vee x = y).$$
  - P must be transitive but not reflexive:
    $$\forall x \forall y \forall z (\neg P(x, x) \wedge ((P(x, y) \wedge P(y, z)) \rightarrow P(x, z))).$$
  - Any two consecutive nodes in P must be adjacent in G:
    $$\forall x \forall y ((P(x, y) \wedge \forall z (\neg P(x, z) \vee \neg P(z, y))) \rightarrow G(x, y)).$$
- It is easy to check that ψ-Graphs is the same as HamiltonPath. Any P with these properties must be a linear order, any two consecutive elements of which are adjacent in G.

Existential Second-Order Expressions and NP

Theorem

For any existential second-order expression $\exists P\varphi$, the problem $\exists P\varphi$-Graphs is in NP.

- Consider a graph $G = (V, E)$ with n nodes.

  If a relation $P^M \subseteq V^{r(P)}$ exists, such that G augmented with P^M satisfies φ, a nondeterministic Turing machine can "guess" such a relation.

  The machine can then go on to test that indeed M satisfies the first-order expression φ deterministically in polynomial time.

  The overall elapsed time for guessing and checking is polynomial, because there are at most $n^{r(P)}$ elements of P^M to guess.

UnReachability vs HamiltonPath

- The expression $\varphi(x,y)$ for UrReachability

$$\begin{aligned}\varphi(x,y) \quad = \quad & \exists P(\forall u \forall v \forall w(P(u,u) \wedge (G(u,v) \to P(u,v)) \\ & \wedge((P(u,v) \wedge P(v,w)) \to P(u,w)) \wedge \neg P(x,y))).\end{aligned}$$

  is in prenex normal form (all quantifiers at the front) with only universal first-order quantifiers, and with matrix in conjunctive normal form.

- More importantly, If we delete from the clauses of the matrix anything that is not an atomic expression involving P, we get:

$$P(u,u), \quad \neg P(x,y), \quad \neg P(u,v) \vee \neg P(v,w) \vee P(u,w).$$

- All three of these clauses have at most one unnegated atomic formula involving P.

UnReachability vs HamiltonPath (Cont'd)

- We call an expression in existential second-order logic a **Horn expression** if
  - it is in prenex form with only universal first-order quantifiers;
  - its matrix is the conjunction of clauses, each of which contains at most one unnegated atomic formula that involves P, the second-order relation symbol.
- In contrast, expression ψ for HamiltonPath contains a host of violations of the Horn form.
  - If it is brought into prenex form there will be existential quantifiers.
  - And $\forall x \forall y (P(x,y) \vee P(y,x) \vee x = y)$ is inherently non-Horn.

Horn Existential Second-Order Expressions and P

Theorem

For any Horn existential second-order expression $\exists P\varphi$, the problem $\exists P\varphi$-GRAPHS is in P.

- Suppose $\exists P\varphi = \exists P\forall x_1 \cdots \forall x_k\eta$, where η is a conjunction of Horn clauses and the arity of P is r.

  Let G be a given graph with vertex set $\{1, 2, \ldots, n\}$.

  The problem is to determine whether G is in $\exists P\varphi$-GRAPHS,

  i.e., whether there exists $P \subseteq \{1, 2 \ldots, n\}^r$, such that φ holds.

  Now we can rewrite $\exists P\varphi$ in the form

  $$\bigwedge_{v_1,\ldots,v_k=1}^{n} \eta[x_1 \leftarrow v_1, \ldots, x_k \leftarrow v_k],$$

  with exactly hn^k clauses, where h is the number of clauses in η.

Proof (Cont'd)

- The atomic expressions in

$$\bigwedge_{v_1,\ldots,v_k=1}^{n} \eta[x_1 \leftarrow v_1, \ldots, x_k \leftarrow v_k]$$

can only be of the forms $G(v_i, v_j)$, $v_i = v_j$ or $P(v_{i_1}, \ldots, v_{i_r})$.

  - The first two kinds can be evaluated in constant time to TRUE or FALSE and disposed of:
    - If a literal is FALSE, it is deleted from a clause;
    - If a literal is TRUE, the clause is deleted;
    - If a clause becomes empty, then G does not satisfy φ.
  - Now we are left with a conjunction of at most hn^k clauses, each of which is a disjunction of atomic expressions of the form $P(v_{i_1}, \ldots, v_{i_r})$ and their negations.

Proof (Conclusion)

- The final step is to realize that each of these expressions can be independently TRUE or FALSE (since we are free to define P as we wish).

  So we may as well replace each by a different Boolean variable, say

  $$P(v_{i_1},\ldots,v_{i_r}) \quad \text{by} \quad x^{v_{i_1},\ldots,v_{i_r}}.$$

  Then we get a Boolean expression F, such that F is satisfiable if and only if there exists P, such that P, taken with G, satisfies φ.

  Because of η's form, F is a Horn Boolean expression with at most hn^k clauses and at most n^r variables.

  But Horn Boolean expressions have a polynomial-time satisfiability problem in their length, and this finishes the proof.

Thank you!

- In closing...

Thank you for your Attention!!