Spanning Trees and the Determinant

George Voutsadakis[1]

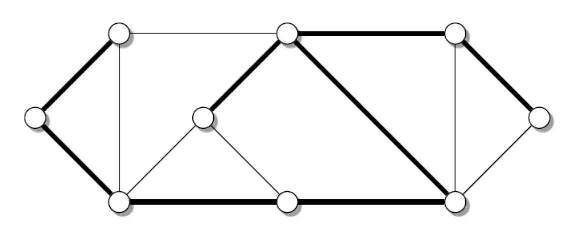
[1]Mathematics and Computer Science
Lake Superior State University

Seminar Presentation

Lake Superior State University

Graphs and Spanning Trees

- A **graph** $G = (V, E)$ consists of
  - A set V of **vertices**;
  - A set E of 2-element subsets of V, called **edges**.



- The **degree** d_i of vertex i is the number of adjacent vertices.
- A **spanning tree** T of G is a subgraph that:
  - **spans** the graph, i.e., that connects all vertices together;
  - is a **tree**, i.e., contains no cycles.

The Problems #SpanningTrees and Determinant

#SpanningTrees: Given a graph G with n vertices, compute the number of spanning trees of G.

- We will see that the number of spanning trees of a graph can be written as the determinant of a version of its adjacency matrix.

Determinant: Given an $n \times n$ matrix A with integer entries, compute its determinant $\det A$.

- Suppose we could show that Determinant is in P, i.e., can be computed in time polynomial in the size of the input.
- Then, given a graph G:
  - We could create the version of the adjacency matrix;
  - Compute its determinant;
  - This would give us the number of spanning trees.

  So #SpanningTrees would also be in P.

The Determinant

- Given a matrix $A = (a_{ij})$, the **determinant of** A is defined by

$$\det A = \sum_{\pi} (-1)^{\pi} \prod_{i=1}^{n} a_{i\pi(i)},$$

  - the sum is over all permutations π of n objects;
  - $(-1)^{\pi}$ is -1 or $+1$ for odd or even permutations, respectively, i.e., those composed of an odd or even number of transpositions (2-element swaps), respectively.
- E.g., the permutations of 2 objects $\{1, 2\}$:

$$\binom{1\ 2}{1\ 2} \text{ is even;}$$

$$\binom{1\ 2}{2\ 1} \text{ is odd.}$$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

3×3 Determinants

- The permutations of 3 objects $\{1, 2, 3\}$:

$$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \text{ are even;}$$
$$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \text{ are odd.}$$

$$\begin{aligned}\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}.\end{aligned}$$

- We can also compute it by reducing to a sum of 3 2×2 determinants, called **expansion in cofactors**:

$$\begin{aligned}\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \\ &\quad + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}\det A^{(1,1)} - a_{12}\det A^{(1,2)} + a_{13}\det A^{(1,3)}.\end{aligned}$$

Properties of the Determinant

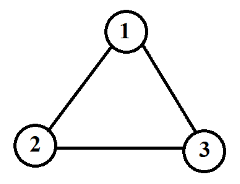
- **Multiplicativity**: $\det(AB) = \det A \cdot \det B$;
- **Linearity w.r.t. Rows**: $\det\begin{bmatrix} r_1 \\ r_2 + ar_2' \\ r_3 \end{bmatrix} = \det\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} + a \cdot \det\begin{bmatrix} r_1 \\ r_2' \\ r_3 \end{bmatrix}$.
  - If two rows are identical, the determinant is zero: $\det\begin{bmatrix} r_1 \\ r_1 \\ r_3 \end{bmatrix} = 0$.
  - Switching rows only changes the sign: $\det\begin{bmatrix} r_3 \\ r_2 \\ r_1 \end{bmatrix} = -\det\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$.
  - Adding a multiple of one row to another row does not change the determinant: $\det\begin{bmatrix} r_1 \\ r_2 + ar_1 \\ r_3 \end{bmatrix} = \det\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$.
- Upper diagonal matrices have determinant equal to the product of their diagonal entries: $\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33}$.

The Laplacian Matrix of a Graph

- Let $G = (V, E)$ be a graph with n vertices.
- The **Laplacian matrix** $L := L(G)$ **of** G is defined by setting, for all $i, j = 1, \ldots, n$,

$$\ell_{ij} = \begin{cases} d_i, & \text{if } i = j \\ -1, & \text{if } \{i, j\} \in E \\ 0, & \text{otherwise} \end{cases}$$

- E.g., consider the graph:



$$L(G) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

- Note that in $L(G)$ each row and each column sums to 0.

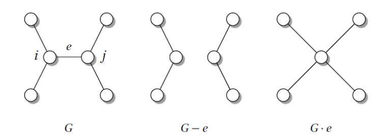
The Matrix-Tree Theorem

- Given a matrix A, we denote by $A^{(i,j)}$ the (i,j)-**minor of** A, i.e., the submatrix obtained from A by deleting the i-th row and the j-th column of A.

The Matrix-Tree Theorem

Let G be an undirected graph and $T(G)$ the number of spanning trees of G. For any i, $T(G) = \det L^{(i,i)}$, where L is the Laplacian of G.

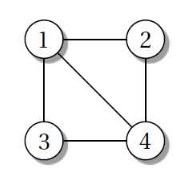
- The proof is by induction on the number of vertices and the number of edges:

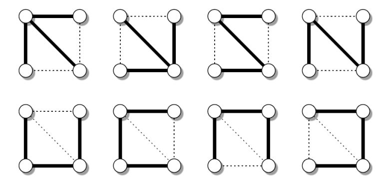


- We note that for an edge e, $T(G) = T(G-e) + T(G \cdot e)$.
- We create the minors of the Laplacians for G, $G-e$ and $G \cdot e$.
- We show that $\det L(G)^{(i,i)} = \det L(G-e)^{(i,i)} + \det L(G \cdot e)^{(j,j)}$.

Example

- Consider the graph on the left below.





- Its Laplacian is

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

- Now we compute

$$T(G) = \det L^{(1,1)} = \det \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix} = 12 - 2 - 2 = 8.$$

Calculation of the Determinant

- The definition of $\det A$ (A an $n \times n$ matrix) involves the sum of $n!$ products ($n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n$ is the number of permutations of n objects).

  So it takes roughly $n!$ time steps.
- Using the expansion in minors ($\det A = \sum_{j=1}^{n}(-1)^{1+j} a_{1j} \det A^{(i,j)}$) requires a sum of n determinants each of size $(n-1) \times (n-1)$.

  So it gives the recursion formula $f(n) = nf(n-1)$.

  So, we get again an estimate for roughly $n!$ time.
- For an efficient calculation, we use the properties of the determinant that we have seen, together with the method of Gaussian elimination.

Example

- Recall the triangulation process using elementary row operations:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 + \frac{1}{2} r_1} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & \frac{5}{2} \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 + \frac{1}{2} r_2} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

- The elementary row operations are effectuated by left multiplication by the corresponding elementary matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

- Since determinants are multiplicative, the product of the determinants of the matrices on the left equals the determinant of the matrix on the right, i.e.,

$$1 \cdot 1 \cdot \det A = 8.$$

The Algorithm

Input: An $n \times n$ matrix A;
Output: $\det A$

For $j = 1$ to $n-1$ (at stage j we cancel a_{ij} for all $i > j$)
If necessary, swap the j-th row with another so that $a_{jj} \neq 0$;
For all $i > j$
$r_i := r_i - \frac{a_{ij}}{a_{jj}} r_j$ (make the new value of a_{ij} is zero);
(Swaps have determinant -1; other moves have determinant 1.)
$\det A$ is the product of the determinants of these moves, and the diagonal entries of the resulting upper-triangular matrix.

- The total number of row operations is at most $n + (n-1) + \cdots + 3 + 2 + 1 = O(n^2)$.
- This algorithm consists of poly(n) arithmetic operations.

  Initially A's entries have poly(n) digits.

  So do all numerators and denominators occurring in the process.

  Thus the total running time is polynomial.

Conclusions

- The problem DETERMINANT is in P.
- The problem #SPANNINGTREES is also in P
- In closing...

Thank you for your Attention!!